An Introduction to the Nonnegative Inverse Eigenvalue Problem with an Algorithm for the RNIEP

by

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1. Introduction

Let $\mathcal{M}_n(\mathbb{S})$ denote the set of $n \times n$ matrices with entries from a set $\mathbb{S}$. An inverse eigenvalue problem aims to reconstruct a matrix $\mathcal{A} \in \mathcal{M}_n(\mathbb{S})$ from prescribed spectral data. Specifically, given a set $\Sigma = \{(\sigma_1, \mu_1), (\sigma_2, \mu_2), ..., (\sigma_k, \mu_k)\}$, such that $\sigma_j \in \mathbb{C}$ and $\mu_j \in \mathbb{N}$, for $1 \leq j \leq k$, we aim to reconstruct a matrix $\mathcal{A} \in \mathcal{M}_n(\mathbb{S})$ that has the numbers σ_j as its eigenvalues, with multiplicity μ_j; such a matrix solves the inverse eigenvalue problem, and we say that $\mathcal{A}$ *realizes* Σ.

The spectral data involved may consist of the complete or only partial information of eigenvalues or eigenvectors [5]. We aim to reconstruct a matrix that fulfills a specific structure as well as that given spectral property. For instance, an inverse eigenvalue problem is always solvable by a matrix $\mathcal{A} \in \mathcal{M}_n(\mathbb{C})$; given a set $\Sigma = \{(\sigma_1, \mu_1), (\sigma_2, \mu_2), ..., (\sigma_k, \mu_k)\}$, one can always construct a diagonal matrix $\mathcal{A} \in \mathcal{M}_n(\mathbb{C})$ with the numbers σ_j, with multiplicity μ_j, along the diagonal. When we impose structural constraints on the realizing matrices, the inverse eigenvalue problem becomes more interesting and complex. Now, it is natural to wonder whether there exists a real matrix $\mathcal{A}$ that realizes Σ - we show that this is also an easy question to answer.

Theorem 1.1. *Given a set $\Sigma = \{(\sigma_1, \mu_1), (\sigma_2, \mu_2), ..., (\sigma_k, \mu_k)\}$, with $\sigma_j \in \mathbb{C}$ and $\sum_{j=1}^{k} \mu_j = n$, there exists $\mathcal{A} \in \mathcal{M}_n(\mathbb{R})$ that realizes Σ if and only if for every pair (σ_j, μ_j) in Σ, the pair $(\bar{\sigma_j}, \mu_j)$, where $\bar{\sigma_j}$ is the complex conjugate of σ_j, is also in Σ.*

Proof. First, suppose that for every pair (σ_j, μ_j) in Σ, the pair $(\bar{\sigma_j}, \mu_j)$, where $\bar{\sigma_j}$ is the complex conjugate of σ_j, is also in Σ. For σ_j nonreal, we write σ_j and $\bar{\sigma_j}$ in exponential form, $\sigma_j = r_j e^{i\theta_j}$ and $\bar{\sigma_j} = r_j e^{-i\theta_j}$, and consider the rotation matrix

$$\mathcal{A}_{\sigma_j} = \begin{pmatrix} r_j \cos(\theta_j) & -r_j \sin(\theta_j) \\ r_j \sin(\theta_j) & r_j \cos(\theta_j) \end{pmatrix}, \tag{1}$$

which has eigenvalues $\lambda_1 = \sigma_j$ and $\lambda_2 = \bar{\sigma_j}$; that is, A_{σ_j} realizes the set $\{(\sigma_j, 1), (\bar{\sigma_j}, 1)\}$. For any σ_j that are real, the 1×1 matrix $[\sigma_j]$ realizes the set $\{(\sigma_j, 1)\}$. Now, using (1) and the fact that the eigenvalues of a block-diagonal matrix are the eigenvalues of each of the diagonal blocks, we construct a block-diagonal matrix $\mathcal{A}$, letting

$$\mathcal{A} = \begin{pmatrix} \boxed{\mathcal{B}_1} & & & & & & & \\ & \boxed{\mathcal{B}_2} & & & & & & \\ & & \boxed{\mathcal{B}_3} & & & & & \\ & & & \ddots & & & & \\ & & & & \boxed{\mathcal{B}_j} & & & \\ & & & & & \boxed{\mathcal{B}_{j+1}} & & \\ & & & & & & \ddots & \\ & & & & & & & \boxed{\mathcal{B}_k} \end{pmatrix}_{n\times n},$$

and such that, for any σ_j nonreal,

$$\mathcal{B}_i = \begin{pmatrix} \mathcal{A}_{\sigma_j} & 0 & 0 & 0 \\ 0 & \mathcal{A}_{\sigma_j} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \mathcal{A}_{\sigma_j} \end{pmatrix}_{2\mu_j\times 2\mu_j},$$

which realizes the set $\{(\sigma_j, \mu_j), (\bar{\sigma_j}, \mu_j)\}$, and for any $\sigma_j \in \mathbb{R}$,

$$\mathcal{B}_i = \begin{pmatrix} \sigma_j & 0 & 0 & 0 \\ 0 & \sigma_j & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sigma_j \end{pmatrix}_{\mu_j\times \mu_j},$$

which realizes the set $\{(\sigma_j, \mu_j)\}$. Thus, by our construction, the matrix $\mathcal{A}$ has real entries and realizes Σ, solving the inverse eigenvalue problem.

Proving the converse, suppose that there exists $\mathcal{A} \in \mathcal{M}_n(\mathbb{R})$ that realizes Σ. Then the characteristic polynomial of $\mathcal{A}$, given by

$$\mathcal{P}_{\mathcal{A}}(\lambda) = \prod_{j=1}^{k} (\lambda - \sigma_j)^{\mu_j},$$

has real coefficients, and thus its nonreal roots come in conjugate pairs. To check that the multiplicities of the nonreal roots and their respective conjugate roots agree, we can write

$$\mathcal{P}_{\mathcal{A}}(\lambda) = (\lambda - \sigma_j)(\lambda - \bar{\sigma_j})\mathcal{Q}^{(1)}(\lambda),$$

where σ_j is an arbitrary, nonreal root of $\mathcal{P}_{\mathcal{A}}(\lambda)$, and $\mathcal{Q}^{(1)}(\lambda)$ is a real polynomial, since

$$(\lambda - \sigma_j)(\lambda - \bar{\sigma_j}) = \lambda^2 - 2\Re(\sigma_j)\lambda + |\sigma_j|^2 \in \mathbb{R}[\lambda].$$

Now if σ_j is not a root of $\mathcal{Q}^{(1)}(\lambda)$, then there is nothing more to prove; otherwise, suppose that σ_j is a root of $\mathcal{Q}^{(1)}(\lambda)$. We may write

$$\mathcal{Q}^{(1)}(\lambda) = (\lambda - \sigma_j)(\lambda - \bar{\sigma_j})\mathcal{Q}^{(2)}(\lambda),$$

where $\mathcal{Q}^{(2)}(\lambda) \in \mathbb{R}[\lambda]$, and thus

$$\mathcal{P}_{\mathcal{A}}(\lambda) = (\lambda - \sigma_j)^2(\lambda - \bar{\sigma_j})^2\mathcal{Q}^{(2)}(\lambda).$$

Repeating this process until σ_j is no longer a root of the polynomial $\mathcal{Q}^{(i)}(\lambda)$, we end up with

$$\mathcal{P}_{\mathcal{A}}(\lambda) = (\lambda - \sigma_j)^{\mu_j}(\lambda - \bar{\sigma_j})^{\mu_j}\mathcal{R}(\lambda), \tag{2}$$

where $\mathcal{R}(\lambda) \in \mathbb{R}[\lambda]$.

Now if $\bar{\sigma_j}$ were a root of $\mathcal{R}(\lambda)$, which is a real polynomial, then σ_j must also be a root of $\mathcal{R}(\lambda)$, which is impossible; and so $\bar{\sigma_j}$ cannot be a root of $\mathcal{R}(\lambda)$. Thus, from (2) we have shown that for every pair (σ_j, μ_j) in Σ, the pair $(\bar{\sigma_j}, \mu_j)$, where $\bar{\sigma_j}$ is the complex conjugate of σ_j, is also in Σ. $\square$

Although we solved the inverse eigenvalue problem with real matrices without any difficulty, in applied mathematics, it is often desirable to impose additional structure on the realizing matrices. In particular, we shall require that the realizing matrices be real, $n \times n$, and with *nonnegative* entries; this sets the stage for us to study the nonnegative inverse eigenvalue problem, henceforth referred to as the NIEP, its history, some results, as well as discuss some open problems.

We shall call elements in the set $\mathcal{M}_n(\mathbb{R}^+)$ *nonnegative matrices*. The NIEP is a problem in which, given a set $\Sigma = \{(\sigma_1, \mu_1), (\sigma_2, \mu_2), ..., (\sigma_k, \mu_k)\}$, with $\sigma_j \in \mathbb{C}$ and $\sum_{j=1}^{k} \mu_k = n$, we aim to reconstruct a matrix $\mathcal{A} \in \mathcal{M}_n(\mathbb{R}^+)$ that realizes Σ. In 1937, Kolmogorov asked when is a given complex number z an eigenvalue of some nonnegative matrix, with Suleimanova [7] extending the question in 1949 to what is now referred to as the NIEP.

Associated with the NIEP are two fundamental questions - the theoretical question of solvability and the practical question of computability. On the theoretical side of the problem, we aim to find necessary and sufficient conditions for which the NIEP has a solution. On the practical side, we want to develop a process for which, knowing a priori the given spectral data, a matrix can be constructed numerically [5]. To this day, both questions of interest remain difficult to answer. At present, the NIEP is open for the case $n \geq 5$.

Because of the wide range of applicability of the NIEP, many subproblems of great interest have developed, e.g., the symmetric nonnegative inverse eigenvalue problem (SNIEP), which aims to reconstruct a symmetric, nonnegative matrix that realizes Σ, and the stochastic inverse eigenvalue probem (StIEP), which aims to reconstruct a stochastic matrix that realizes Σ.

This thesis is organized as follows. In Section 2, we discuss the solvability of the NIEP for the case $n = 2$ and compute the realizing matrices explicitly. In Section 3, we present some important theorems that help towards solving the NIEP for generalized n. The main theorems that will be used in this thesis are the Perron-Frobenius theorem, which tells us that every nonnegative $n \times n$ matrix has a dominant, nonnegative eigenvalue with a corresponding entrywise-nonnegative eigenvector, and a theorem by Nazari and Sherafat [6] for combining eigenvalues of two nonnegative matrices. We then use these theorems to develop an algorithm [4] for solving the real nonnegative inverse eigenvalue problem (RNIEP). In Section 4, we investigate the solvability of the NIEP for the case $n = 3$. Concluding remarks are given in Section 5.

2. Solvability of the NIEP, for $n = 2$

We begin our study of the NIEP for small n; specifically, we consider the case $n = 2$ and investigate the solvability of the problem. If a solution exists, we aim to reconstruct the realizing matrix.

Theorem 2.1. *Given a set* $\Sigma = \{(\sigma_1, \mu_1), (\sigma_2, \mu_2), ..., (\sigma_k, \mu_k)\}$, *with* $\sum_{j=1}^{k} \mu_j = 2$, *there exists* $\mathcal{A} \in \mathcal{M}_2(\mathbb{R}^+)$ *that realizes* Σ *if and only if* σ_1 *and* $\sigma_2 \in \mathbb{R}$ *and, up to ordering,* $0 \leq |\sigma_2| \leq \sigma_1$.

Proof. First, suppose that there exists $\mathcal{A} \in \mathcal{M}_2(\mathbb{R}^+)$ that realizes Σ and denote the matrix as

$$\mathcal{A} = \begin{pmatrix} \alpha & \beta \\ \\ \gamma & \mu \end{pmatrix},$$

with $\alpha, \beta, \gamma, \mu \geq 0$. Since $\mathcal{A}$ is real, by Theorem 1.1, σ_1 and σ_2 must be either a complex conjugate pair or they are both real. Suppose they are a complex conjugate pair and let $\sigma_1 = \xi + i\phi$ and $\sigma_2 = \xi - i\phi$, with $\phi \neq 0$. As the determinant of $\mathcal{A}$ is equal to the product of its eigenvalues and the trace of $\mathcal{A}$ is equal to the sum of its eigenvalues, we have the relations

$$det(\mathcal{A}) = \alpha\mu - \beta\gamma = \sigma_1\sigma_2 = (\xi + i\phi)(\xi - i\phi) = \xi^2 + \phi^2 \tag{3}$$

$$tr(\mathcal{A}) = \alpha + \mu = \sigma_1 + \sigma_2 = (\xi + i\phi) + (\xi - i\phi) = 2\xi. \tag{4}$$

Now we may multiply (4) by μ and subtract it from (3), which gives us the relation

$$\begin{aligned} -\beta\gamma - \mu^2 &= \xi^2 + \phi^2 - 2\xi\mu \\ \implies -\beta\gamma &= \xi^2 + \phi^2 - 2\xi\mu + \mu^2 \\ \implies -\beta\gamma &= (\xi - \mu)^2 + \phi^2. \end{aligned} \tag{5}$$

Since we assumed that $\phi \neq 0$, then the right-hand side of (5) is strictly positive, which implies that either β or $\gamma < 0$, which is a contradiction. Thus, σ_1 and σ_2 are both real.

The characteristic polynomial of $\mathcal{A}$ is given by

$$\mathcal{P}_{\mathcal{A}}(\lambda) = \det[\mathcal{A} - \lambda\mathcal{I}] = (\alpha - \lambda)(\mu - \lambda) - \beta\gamma = \lambda^2 - (\alpha + \mu)\lambda + (\alpha\mu - \beta\gamma),$$

and since σ_1 and σ_2 are real roots of $\mathcal{P}_{\mathcal{A}}(\lambda)$, they have the form (up to ordering)

$$\sigma_1 = \frac{(\alpha + \mu) + \sqrt{(\alpha + \mu)^2 - 4(\alpha\mu - \beta\gamma)}}{2} \tag{6}$$

$$\sigma_2 = \frac{(\alpha + \mu) - \sqrt{(\alpha + \mu)^2 - 4(\alpha\mu - \beta\gamma)}}{2} \tag{7}$$

with $(\alpha+\mu)^2 - 4(\alpha\mu - \beta\gamma) \geq 0$. Moreover, $(\alpha+\mu) \geq 0$ by assumption. If $(\alpha+\mu)^2 - 4(\alpha\mu - \beta\gamma) = 0$, then $\Sigma = \{(\sigma_1, 2)\}$, with $\sigma_1 = \frac{\alpha+\mu}{2}$, which implies that $0 \leq \sigma_1$. Otherwise, $(\alpha+\mu)^2 - 4(\alpha\mu - \beta\gamma) > 0$

and $\Sigma = \{(\sigma_1, 1), (\sigma_2, 1)\}$. Adding (6) and (7) gives

$$\sigma_1 + \sigma_2 = (\alpha + \mu) \geq 0, \tag{8}$$

and subtracting (7) from (6) gives

$$\sigma_1 - \sigma_2 = \sqrt{(\alpha + \mu)^2 - 4(\alpha\mu - \beta\gamma)} \geq 0. \tag{9}$$

Thus, from (8) and (9) we have that

$$-\sigma_1 \leq \sigma_2 \leq \sigma_1, \tag{10}$$

which implies that $0 \leq |\sigma_2| \leq \sigma_1$.

Proving the converse, suppose that (up to ordering) σ_1 and $\sigma_2 \in \mathbb{R}$ and $0 \leq |\sigma_2| \leq \sigma_1$. If the σ_i are not distinct, then $\Sigma = \{(\sigma_1, 2)\}$ and by assumption, $0 \leq \sigma_1$. Thus, the diagonal matrix

$$\mathcal{A} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}$$

is nonnegative and realizes Σ. Otherwise, suppose that the σ_i are distinct, so that $\Sigma = \{(\sigma_1, 1), (\sigma_2, 1)\}$. We take a constructive approach to show the existence of the realizing matrix $\mathcal{A}$ and compute it explicitly. To this end, we consider the matrix

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

and try to reconstruct the entries $a_{11}, a_{12}, a_{21}, a_{22}$ such that the relations

$$tr(\mathcal{A}) = \sigma_1 + \sigma_2 = a_{11} + a_{22} \tag{11}$$

$$det(\mathcal{A}) = \sigma_1\sigma_2 = a_{11}a_{22} - a_{12}a_{21} \tag{12}$$

are fulfilled. If we are successful in doing so, then $\mathcal{A}$ will have σ_1 and σ_2 as its eigenvalues.

From (11), a suitable choice for the entries a_{11} and a_{22} is

$$a_{11} = a_{22} = \frac{\sigma_1 + \sigma_2}{2},$$

which gives the matrix

$$\mathcal{A} = \begin{pmatrix} \frac{\sigma_1+\sigma_2}{2} & a_{12} \\ \\ a_{21} & \frac{\sigma_1+\sigma_2}{2} \end{pmatrix}.$$

Now with the entries a_{11} and a_{22} fixed, we consider the second relation (12), which becomes

$$\det(\mathcal{A}) = \sigma_1\sigma_2 = a_{11}a_{22} - a_{12}a_{21} = \frac{(\sigma_1+\sigma_2)^2}{4} - a_{12}a_{21}$$

$$\implies \left(\frac{\sigma_1-\sigma_2}{2}\right)^2 = a_{12}a_{21},$$

and we may let

$$a_{12} = a_{21} = \frac{\sigma_1-\sigma_2}{2}.$$

Thus, by our construction, the matrix

$$\mathcal{A} = \begin{pmatrix} \frac{\sigma_1+\sigma_2}{2} & \frac{\sigma_1-\sigma_2}{2} \\ \\ \frac{\sigma_1-\sigma_2}{2} & \frac{\sigma_1+\sigma_2}{2} \end{pmatrix}$$

has real entries and realizes Σ. Moreover, $\mathcal{A}$ is symmetric, and from the assumption that $0 \leq |\sigma_2| \leq \sigma_1$, we see that the entries of $\mathcal{A}$ are nonnegative, and this completes the proof. $\square$

Corollary. *For n=2, the symmetric nonnegative inverse eigenvalue problem (SNIEP) has a solution if and only if the nonnegative inverse eigenvalue problem (NIEP) has a solution.*

For the next corollary [1], we study a doubly-stochastic inverse eigenvalue problem and provide necessary and sufficient conditions for there to exist a doubly-stochastic matrix that realizes Σ. Recall that a doubly-stochastic matrix has nonnegative entries, with the additional structure that the rows and columns each sum to one; thus, any doubly-stochastic matrix is of the form

$$\begin{pmatrix} 1-\zeta & \zeta \\ \zeta & 1-\zeta \end{pmatrix},$$

with $\zeta \in [0,1]$.

Corollary. *Given a set* $\Sigma = \{(\sigma_1, \mu_1), (\sigma_2, \mu_2), ..., (\sigma_k, \mu_k)\}$*, with* $\sum_{j=1}^{k} \mu_j = 2$*, there exists a doubly-stochastic matrix* $\mathcal{A} \in \mathcal{M}_2(\mathbb{R}^+)$ *that realizes* Σ *if and only if, up to ordering,* $\sigma_1 = 1$ *and* $\sigma_2 \in \mathbb{R}$*, with* $|\sigma_2| \leq 1$*.*

Proof. First, suppose that there exists a doubly-stochastic matrix $\mathcal{A} \in \mathcal{M}_2(\mathbb{R}^+)$ that realizes Σ and let the matrix be

$$\mathcal{A} = \begin{pmatrix} 1-\zeta & \zeta \\ \zeta & 1-\zeta \end{pmatrix},$$

for $\zeta \in [0,1]$. Its characteristic polynomial is

$$\mathcal{P}_{\mathcal{A}}(\lambda) = (1-\zeta-\lambda)^2 - \zeta^2 = (1-\lambda)(1-\lambda-2\zeta),$$

which has roots $\sigma_1 = 1$ and $\sigma_2 = 1 - 2\zeta$.

If $\zeta = 0$, then $\Sigma = \{(\sigma_1, 2)\}$, with $\sigma_1 = 1$. If $\zeta \neq 0$, then $\Sigma = \{\sigma_1, 1), (\sigma_2, 1)\}$ with $\sigma_1 = 1$. To check σ_2, we solve for ζ, which gives $\zeta = \frac{1-\sigma_2}{2}$, and thus the matrix

$$\mathcal{A} = \begin{pmatrix} 1 - \frac{1-\sigma_2}{2} & \frac{1-\sigma_2}{2} \\ \frac{1-\sigma_2}{2} & 1 - \frac{1-\sigma_2}{2} \end{pmatrix}$$

realizes Σ. As $\mathcal{A}$ is doubly-stochastic, it has nonnegative entries, which forces σ_2 to be real and that

$$0 \leq \frac{1-\sigma_2}{2} \leq 1,$$

which implies that $|\sigma_2| \leq 1$.

Proving the converse, suppose that $\Sigma = \{(\sigma_1, 2)\}$ with $\sigma_1 = 1$. Then the identity matrix

$$\mathcal{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is doubly-stochastic and realizes Σ. Otherwise, $\Sigma = \{(\sigma_1, 1), (\sigma_2, 1)\}$, and (up to ordering) $\sigma_1 = 1$ and $\sigma_2 \in \mathbb{R}$, with $|\sigma_2| \leq 1$. We apply Theorem 2.1, and thus

$$\mathcal{A} = \begin{pmatrix} \frac{1+\sigma_2}{2} & \frac{1-\sigma_2}{2} \\ \frac{1-\sigma_2}{2} & \frac{1+\sigma_2}{2} \end{pmatrix},$$

which is clearly doubly-stochastic, realizes Σ. $\square$

3. Towards solving the NIEP, for generalized n

In this section, we give some important results that are used to understand the NIEP in full generality. We start with proving the so-called JLL inequality ([2], [9]).

Theorem 3.1. *Given a set $\Sigma = \{(\sigma_1, \mu_1), (\sigma_2, \mu_2), ..., (\sigma_k, \mu_k)\}$ and $\mathcal{A} \in \mathcal{M}_n(\mathbb{R}^+)$, suppose that $\mathcal{A}$ realizes Σ. We define the r^{th} moment $s_r(\Sigma)$ of Σ by*

$$s_r(\Sigma) = \sum_{j=1}^{k} \mu_j \sigma_j^r = tr(\mathcal{A}^r), \quad r = 1, 2, ... \tag{13}$$

Then

$$[s_r(\Sigma)]^m \leq n^{m-1} s_{rm}(\Sigma), \quad r, m = 1, 2, ... \tag{14}$$

Proof. We decompose the matrix $\mathcal{A}$ into $\mathcal{A} = \mathcal{C} + \mathcal{D}$ such that the matrix $\mathcal{D}$ is diagonal, with the diagonal entries a_{ii} of $\mathcal{A}$ along the diagonal of $\mathcal{D}$, and thus the matrix $\mathcal{C}$ has the off-diagonal entries $(a_{ij}, i \neq j)$ of $\mathcal{A}$ and $0s$ along its diagonal. Now since $\mathcal{A}$ is nonnegative, then so is $\mathcal{C} + \mathcal{D}$; moreover, the matrices $\mathcal{A}^m, \mathcal{C}^m$, and $\mathcal{D}^m$ are all nonnegative for $m \in \mathbb{N}$. Then it follows that

$$\mathcal{A}^m = (\mathcal{C} + \mathcal{D})^m = (\mathcal{C} + \mathcal{D})(\mathcal{C} + \mathcal{D})...(\mathcal{C} + \mathcal{D}) = \mathcal{C}^m + \mathcal{M} + \mathcal{D}^m,$$

where $\mathcal{M}$ is a nonnegative matrix. This implies that $\mathcal{A}^m - \mathcal{C}^m - \mathcal{D}^m$ is a nonnegative matrix. Thus, we have that

$$tr\left(\mathcal{A}^m - \mathcal{C}^m - \mathcal{D}^m\right) = tr(\mathcal{A}^m) - tr(\mathcal{C}^m) - tr(\mathcal{D}^m) \geq 0$$

$$\implies tr(\mathcal{A}^m) - tr(\mathcal{D}^m) \geq tr(\mathcal{C}^m)$$

$$\implies tr(\mathcal{A}^m) - \sum_{i=1}^{n} a_{ii}^m \geq tr(\mathcal{C}^m)$$

$$\implies s_m(\Sigma) - \sum_{i=1}^{n} a_{ii}^m \geq tr(\mathcal{C}^m). \tag{15}$$

Now, for $p = m$ and $q = \frac{m}{m-1}$, we have that $\frac{1}{p} + \frac{1}{q} = 1$, and we can apply Holder's inequality to observe that

$$\sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} a_{ii} \times 1 \leq \left(\sum_{i=1}^{n} a_{ii}^m\right)^{\frac{1}{m}} \left(\sum_{i=1}^{n} 1^{\frac{m}{m-1}}\right)^{\frac{m-1}{m}} = \left(\sum_{i=1}^{n} a_{ii}^m\right)^{\frac{1}{m}} \left(n^{\frac{m-1}{m}}\right).$$

$$\implies \left(\sum_{i=1}^{n} a_{ii}\right)^m = [tr(\mathcal{A})]^m = [s_1(\Sigma)]^m \leq n^{m-1} \sum_{i=1}^{n} a_{ii}^m. \tag{16}$$

Multiplying (15) by n^{m-1} and using (16), it follows that

$$n^{m-1} s_m(\Sigma) - n^{m-1} \sum_{i=i}^{n} a_{ii}^m \geq n^{m-1} tr(\mathcal{C}^m) \geq 0$$

$$\implies n^{m-1} s_m(\Sigma) - [s_1(\Sigma)]^m \geq 0$$

$$\implies [s_1(\Sigma)]^m \leq n^{m-1} s_m(\Sigma). \tag{17}$$

Finally, we may apply (17) to the nonnegative matrix $\mathcal{A}^r$ to obtain

$$[s_r(\Sigma)]^m \leq n^{m-1} s_{rm}(\Sigma), \quad r, m = 1, 2, ...$$

as desired. □

We now prove Perron's theorem for entrywise positive $n \times n$ matrices - henceforth referred to as *positive matrices* - and the Perron-Frobenius theorem for nonnegative $n \times n$ matrices [3]. We will write $v \geq 0$ or $v > 0$ for vectors v with nonnegative or positive components, respectively. Moreover, when we write a vector inequality for vectors in $\mathcal{R}^n$, it means that the inequality holds for all corresponding components. We begin with a lemma.

Lemma 3.2. *Let $\mathcal{A}$ be an $n \times n$ matrix. Denote by $p(\mathcal{A})$ to be the set of all nonnegative numbers λ for which there is a nonnegative vector $x \neq 0$ such that*

$$\mathcal{A}x \geq \lambda x, \quad x \geq 0. \tag{18}$$

Then for $\mathcal{A}$ positive,

(i) $p(\mathcal{A})$ is nonempty and contains a positive number,

(ii) $p(\mathcal{A})$ is bounded,

(iii) $p(\mathcal{A})$ is closed.

Proof. Consider an arbitrary positive vector x; since $\mathcal{A}$ is positive, $\mathcal{A}x$ is a positive vector. Then (18) will hold by taking $\lambda > 0$ and sufficiently small, proving (i) of the lemma.

Now since both sides of (18) are linear in x, we may normalize x so that

$$\zeta x = \sum x_i = 1, \quad \zeta = (1, ..., 1).$$

Then multiplying (18) by ζ on the left gives us:

$$\zeta \mathcal{A} x \geq \lambda \zeta x = \lambda. \tag{19}$$

Denote the largest component of $\zeta\mathcal{A}$ by b; then we have that $b\zeta \geq \zeta\mathcal{A}$. Applying this to (19), we have that

$$b\zeta x \geq \lambda \zeta x = \lambda$$

$$\implies b \geq \lambda,$$

which shows that the set $p(\mathcal{A})$ is bounded, proving part (ii) of the lemma.

To prove part (iii) of the lemma, consider a convergent sequence λ_n, converging to λ, in the set $p(\mathcal{A})$; by definition there is a corresponding $x_n \neq 0$ such that (18) holds:

$$\mathcal{A}x_n \geq \lambda_n x_n, \quad x_n \geq 0. \tag{20}$$

We may assume that the x_n are normalized:

$$\zeta x_n = 1.$$

Now, the set of nonnegative x_n, normalized, is a closed and bounded set in $\mathcal{R}^n$, and thus it is compact. Then a subsequence of x_n converges to a nonnegative x, also normalized, while λ_n converges to λ; and passing to the limit of (20), we have that x, λ satisfy (18). Therefore, the set $p(\mathcal{A})$ is closed, proving part(iii) of the lemma. □

Theorem 3.3. *(Perron's theorem) Every positive $n \times n$ matrix $\mathcal{A}$ has a dominant eigenvalue, denoted by $\sigma(\mathcal{A})$, which has the following properties:*

(i) $\sigma(\mathcal{A})$ is real and positive, and it has an associated eigenvector h with positive entries:

$$\mathcal{A}h = \sigma(\mathcal{A})h, \quad h > 0.$$

(ii) $\sigma(\mathcal{A})$ is a simple eigenvalue.
(iii) Every other eigenvalue σ_j of $\mathcal{A}$ is strictly less than $\sigma(\mathcal{A})$ in absolute value:

$$|\sigma_j| < \sigma(\mathcal{A}).$$

(iv) For every other eigenvalue σ_j of $\mathcal{A}$, there does not exist an associated eigenvector f such that all its components are nonnegative.

Proof. Let $\mathcal{A}$ be a positive $n \times n$ matrix. The set $p(\mathcal{A})$ from Lemma 3.2 is closed and bounded, and thus it has a maximum λ_{max}. By part(i) of Lemma 3.2, we know that $\lambda_{max} > 0$. We will show that λ_{max} is the dominant eigenvalue of $\mathcal{A}$.

First, we show that λ_{max} is an eigenvalue of $\mathcal{A}$. Since (18) is satisfied by λ_{max}, there is a nonnegative vector h such that

$$\mathcal{A}h \geq \lambda_{max} h, \quad h \geq 0, h \neq 0. \tag{21}$$

We show that equality holds in (21); suppose, for contradiction, in the kth component:

$$\sum a_{ij} h_j \geq \lambda_{max} h_i, \quad i \neq k \tag{22}$$

$$\sum a_{kj} h_j > \lambda_{max} h_k. \tag{23}$$

Define the vector $x = h + \epsilon e_k$, where $\epsilon > 0$ and e_k has kth component equal to 1, and all other components equal to zero. Now since $\mathcal{A}$ is positive, replacing h by x in (21) increases each component of the left-hand side: $\mathcal{A}x > \mathcal{A}h$, while on the right-hand side, only the kth component is increased when h is replaced by x. Then it follows from (22) and (23) that, for $\epsilon > 0$ and sufficiently small, we have that

$$\mathcal{A}x > \lambda_{max} x, \tag{24}$$

and since this inequality is strict, there exists $\delta > 0$ and sufficiently small such that

$$\mathcal{A}x > (\lambda_{max} + \delta)x. \tag{25}$$

This implies that $(\lambda_{max} + \delta)$ belongs to the set $p(\mathcal{A})$, which contradicts the fact that λ_{max} is the maximum in the set. Thus, (21) must be an equality, and this proves that λ_{max} is an eigenvalue of $\mathcal{A}$, with a corresponding eigenvector h that is nonnegative. We now show that the eigenvector h is, in fact, positive. Since $\mathcal{A}$ is positive and $h \geq 0$, $h \neq 0$, then it is clear that $\mathcal{A}h > 0$. But since $\mathcal{A}h = \lambda_{max} h$, and since λ_{max} is positive, then this forces h to be positive, which proves part(i) of Theorem 3.3.

Next, we prove that λ_{max} is a simple eigenvalue. We claim that all eigenvectors of $\mathcal{A}$ with eigenvalue λ_{max} must be proportional to h; suppose, for contradiction, that there exists another eigenvector y with eigenvalue λ_{max} that is not a scalar-multiple of h. Then we may consider the eigenvector $h + cy$, with c chosen suitably so that $h + cy \geq 0$ but that one of the components of $h + cy$ is zero; but this contradicts the fact that an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue λ_{max} is not only nonnegative but also positive.

To complete the proof of part(ii), we show that $\mathcal{A}$ has no generalized eigenvectors for the eigenvalue λ_{max}; that is, a vector y such that

$$\mathcal{A}y = \lambda_{max}y + ch. \tag{26}$$

In (26) we may replace y with $-y$ if necessary to ensure that $c > 0$; and by replacing y with $y + bh$ if necessary we can make sure that y is positive. Then it follows from (26), and the fact that $h > 0$, that $\mathcal{A}y > \lambda_{max}y$, which implies that there exists $\delta > 0$ and sufficiently small such that

$$\mathcal{A}y > (\lambda_{max} + \delta)y.$$

This shows that $(\lambda_{max} + \delta)$ is in the set $p(\mathcal{A})$, which contradicts the fact that λ_{max} is the maximum of $p(\mathcal{A})$. Thus, there are no generalized eigenvectors y with corresponding eigenvalue λ_{max}, and this completes the proof of part(ii) of Theorem 3.3.

To prove part(iii) of Theorem 3.3, let κ be another eigenvalue of $\mathcal{A}$ that is not equal to λ_{max}, with y a corresponding eigenvector, so that

$$\mathcal{A}y = \kappa y$$

(here κ and the components of y may be complex numbers.)
Then we have that the i^{th} component of the vector $\mathcal{A}y$ is given by

$$\sum_j a_{ij}y_j = \kappa y_i,$$

and by the triangle inequality, we have that

$$\sum_j a_{ij}|y_j| \geq \left|\sum_j a_{ij}y_j\right| = |\kappa y_i| = |\kappa|\,|y_i|\,, \tag{27}$$

which, by comparison with (18) of Lemma 3.2, shows that $|\kappa|$ is in the set $p(\mathcal{A})$.

Now suppose that $|\kappa| = \lambda_{max}$. Then we have that the vector

$$y = (|y_1|, |y_2|, ..., |y_n|)$$

is an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue λ_{max}, and so it must be proportional to h, giving us

$$|y_i| = ch_i. \tag{28}$$

Moreover, the first inequality in (27) becomes an equality – and it is easy to show that this *triangle equality* holds if and only if all the components y_i have the same complex argument, so that

$$y_i = |y_i|e^{i\theta}.$$

Now combining this with (28), we get that

$$y_i = ch_i e^{i\theta}$$

$$\implies y = ce^{i\theta}h$$

$$\implies \mathcal{A}y = \kappa y = \mathcal{A}(ce^{i\theta})h = (ce^{i\theta})\mathcal{A}h = ce^{i\theta}\lambda_{max}h = ce^{i\theta}\lambda_{max}\frac{1}{ce^{i\theta}}y = \lambda_{max}y.$$

Thus, we see that $\kappa = \lambda_{max}$, which proves part(iii) of Theorem 3.3.

To prove part(iv), we recall that the inner product $< \cdot, \cdot >$ of eigenvectors of $\mathcal{A}$ and $\mathcal{A}^t$ for distinct, real eigenvalues is zero, since for an eigenvector x of $\mathcal{A}$ with real eigenvalue a and for an eigenvector y of $\mathcal{A}^t$ with real eigenvalue b, we know that

$$< \mathcal{A}x, y > = < x, \mathcal{A}^t y >$$

$$\implies < ax, y > = < x, by >$$

$$\implies a < x, y > = \bar{b} < x, y >, \tag{29}$$

which shows that $< x, y > = 0$, if $a \neq b$. Now since $\mathcal{A}$ is assumed to be positive, then $\mathcal{A}^t$ is also positive; moreover, as $\mathcal{A}$ and $\mathcal{A}^t$ are similar, we know that they have the same eigenvalues. In particular, $\mathcal{A}^t$ has the dominant eigenvalue $\sigma(\mathcal{A})$ – and we know that this dominant eigenvalue has a corresponding positive eigenvector ψ. Now, consider any other eigenvalue σ_j, different from the dominant eigenvalue, with corresponding eigenvector f. Using the fact from (29), since the positive eigenvector ψ cannot annihilate a *nonnegative* - and nonzero - eigenvector f, then there cannot possibly be another nonnegative eigenvector f corresponding to an eigenvalue that's different from the dominant eigenvalue, and this completes the proof of Theorem 3.3. $\square$

Theorem 3.4. *(Perron-Frobenius theorem) Every nonnegative $n \times n$ matrix $\mathcal{A}$, $\mathcal{A} \neq 0$, has a dominant eigenvalue $\sigma(\mathcal{A})$ with the following properties:*

(i) $\sigma(\mathcal{A})$ is real and nonnegative, and has a corresponding eigenvector h with nonnegative entries:

$$\mathcal{A}h = \sigma(\mathcal{A})h, \quad h \geq 0 \tag{30}$$

(ii) Every other eigenvalue κ is less than or equal to $\sigma(\mathcal{A})$ in absolute value:

$$|\kappa| \leq \sigma(\mathcal{A}). \tag{31}$$

Proof. Let $\mathcal{A}$ be a nonnegative $n \times n$ matrix. We approximate $\mathcal{A}$ by a convergent sequence $\mathcal{A}_n$ of entrywise-positive matrices. Since the characteristic polynomials of $\mathcal{A}_n$ converge to the characteristic polynomial of $\mathcal{A}$, we know that the eigenvalues of $\mathcal{A}_n$ converge to the eigenvalues of $\mathcal{A}$. Now define

$$\sigma(\mathcal{A}) = \lim_{n\to\infty} \sigma(\mathcal{A}_n), \tag{32}$$

where, just as in Perron's theorem, $\sigma(\mathcal{A})$ denotes the dominant eigenvalue of $\mathcal{A}$ and, similarly, $\sigma(\mathcal{A}_n)$ denotes the dominant eigenvalue of $\mathcal{A}_n$.

To prove part(i), we use the dominant eigenvector h_n of $\mathcal{A}_n$, corresponding to the dominant eigenvalue $\sigma(\mathcal{A}_n)$, normalized:

$$\zeta h_n = 1, \quad \zeta = (1, 1, ...1).$$

This finite set of n eigenvectors is trivially compact, and so a subsequence of h_n converges to a limit vector h, while $\sigma(\mathcal{A}_n)$ converges to $\sigma(\mathcal{A})$, as defined in (32). Since h is the limit of normalized positive vectors, it is a nonnegative vector. Moreover, each h_n satisfies

$$\mathcal{A}_n h_n = \sigma(\mathcal{A}_n)h_n. \tag{33}$$

Now as $n \to \infty$ we obtain the relation (30), which proves part(i).

From (33), as $n \to \infty$, we may apply part(iii) of Perron's theorem to $\mathcal{A}_n$, and the inequality in part(ii) of Theorem 3.4 is immediate. $\square$

An algorithm for the RNIEP: Now we give an algorithm to solve the RNIEP; this method is due to the recent work of Lin [4]. After developing this method, we demonstrate the algorithm with an example. First, we prove a useful theorem by Nazari and Sherafat [6] for combining the eigenvalues of two nonnegative matrices.

Theorem 3.5. *Suppose that* $\mathcal{A} \in \mathcal{M}_n(\mathbb{R}^+)$ *realizes* $\Sigma = \{(\sigma_1, \mu_1), (\sigma_2, \mu_2), ..., (\sigma_k, \mu_k)\}$*, where* $\sigma_j \in \mathbb{C}$ *and* $\sum_{j=1}^{k} \mu_j = n$*,* $\mathcal{B} \in \mathcal{M}_m(\mathbb{R}^+)$ *realizes* $\Gamma = \{(\gamma_1, \nu_1), (\gamma_2, \nu_2), ..., (\gamma_k, \nu_k)\}$*, where* $\gamma_j \in \mathbb{C}$ *and* $\sum_{j=1}^{k} \nu_j = m$*, and such that* $\sigma_1 \geq |\sigma_j|$ *and* $\gamma_1 \geq |\gamma_j|$ *for* $j > 1$*. Let* v *be a normalized eigenvector corresponding to the dominant eigenvalue* γ_1 *of the matrix* $\mathcal{B}$*. If* $\mathcal{A}$ *has the form*

$$\begin{pmatrix} \mathcal{A}_1 & a \\ b^t & \gamma_1 \end{pmatrix},$$

where $\mathcal{A}_1 \in \mathcal{M}_{n-1}(\mathbb{R}^+)$*, and* a *and* b *are two vectors in* $\mathbb{R}^{n-1}$*, then* $\mathcal{C} \in \mathcal{M}_{n+m-1}(\mathbb{R}^+)$ *having the form*

$$\begin{pmatrix} \mathcal{A}_1 & av^t \\ vb^t & \mathcal{B} \end{pmatrix}$$

realizes the set $\Sigma \cup \Gamma' = \{(\sigma_1, \mu_1), (\sigma_2, \mu_2), ..., (\sigma_k, \mu_k)\} \bigcup \{(\gamma_1, \nu_1 - 1), (\gamma_2, \nu_2), (\gamma_3, \nu_3), ..., (\gamma_k, \nu_k)\}$*.*

Proof. Starting with v, which by assumption is a normalized eigenvector corresponding to the dominant eigenvalue γ_1 of the nonnegative matrix $\mathcal{B}$, we can compute (by using the Gram-Schmidt algorithm and then normalizing) an orthonormal basis $\{v, w_1, ..., w_{m-1}\}$ for $\mathbb{C}^m$. Let $\mathcal{W}_1$ be an $m \times (m-1)$ matrix whose columns are the orthonormal basis vectors w_1,...,w_{n-1}. Then the augmented matrix $\mathcal{Y}_1 = (v \quad \mathcal{W}_1)$ is unitary. Thus, we have that

$$\mathcal{B}\mathcal{Y}_1 = \mathcal{B}\,(v \quad \mathcal{W}_1) = (\mathcal{B}v \quad \mathcal{B}\mathcal{W}_1) = (\gamma_1 v \quad \mathcal{B}\mathcal{W}_1)\,,$$

and using the inverse $\mathcal{Y}_1^*$, we have that

$$\mathcal{Y}_1^* \mathcal{B} \mathcal{Y}_1 = \begin{pmatrix} v* \\ \mathcal{W}_1^* \end{pmatrix} \begin{pmatrix} \gamma_1 v & \mathcal{B}\mathcal{W}_1 \end{pmatrix} = \begin{pmatrix} v^*\gamma_1 v & v^*\mathcal{B}\mathcal{W}_1 \\ \mathcal{W}_1^*\gamma_1 v & \mathcal{W}_1^*\mathcal{B}\mathcal{W}_1 \end{pmatrix} = \begin{pmatrix} \gamma_1 v^* v & v^*\mathcal{B}\mathcal{W}_1 \\ \gamma_1 \mathcal{W}_1^* v & \mathcal{W}_1^*\mathcal{B}\mathcal{W}_1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_1 & v^*\mathcal{B}\mathcal{W}_1 & ... \\ 0 & & \\ \vdots & \hat{\mathcal{B}} & \\ 0 & & \end{pmatrix} =: \mathcal{B}_1,$$

where the matrix $\mathcal{B}_1$ is gotten from the fact that $v^* v = 1$, $\mathcal{W}_1^* v = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ from orthogonality of the vectors in $\{v, w_1, ..., w_{m-1}\}$, and the entry $\mathcal{W}_1^* \mathcal{B} \mathcal{W}_1 = \hat{\mathcal{B}}$, which is an $(m-1)\times(m-1)$ block. Now since $\mathcal{B}$ is similar to $\mathcal{B}_1$, and $\mathcal{B}$ realizes $\Gamma = \{(\gamma_1, \nu_1), (\gamma_2, \nu_2), ..., (\gamma_k, \nu_k)\}$ by assumption, then $\mathcal{B}_1$ realizes Γ, too. As the characteristic polynomial of $\mathcal{B}_1$ is $\mathcal{P}_{\mathcal{B}_1}(\gamma) = \det[\mathcal{B}_1 - \gamma\mathcal{I}]$, which is easily computed by expansion by minors from expanding along the first column of $\mathcal{B}_1$, we see that

$$\mathcal{P}_{\mathcal{B}_1}(\gamma) = \det[\mathcal{B}_1 - \gamma\mathcal{I}] = (\gamma_1 - \gamma)\det\left[\hat{B} - \gamma\mathcal{I}\right],$$

and it follows that the block $\hat{\mathcal{B}}$ realizes the set $\Gamma' = \{(\gamma_1, \nu_1 - 1), (\gamma_2, \nu_2), ..., (\gamma_k, \nu_k)\}$. Now, applying the Shur decomposition theorem, there exists an $(m-1)\times(m-1)$ unitary matrix $\mathcal{W}_2$ such that $\mathcal{W}_2^* \hat{\mathcal{B}} \mathcal{W}_2 = \hat{\mathcal{T}}_{\mathcal{B}}$, where $\hat{\mathcal{T}}_{\mathcal{B}}$ is an upper-triangular matrix with the eigenvalues $\gamma_2, ..., \gamma_k$, counting multiplicity, on the diagonal.

Now define

$$\mathcal{Y}_2 = \begin{pmatrix} 1 & 0... & 0 \\ 0 & & \\ \vdots & \mathcal{W}_2 & \\ 0 & & \end{pmatrix}.$$

As $\mathcal{W}_2$ is unitary, $\mathcal{Y}_2$ is also unitary, and we have that

$$\mathcal{Y}_2^* \mathcal{B}_1 \mathcal{Y}_2 = \mathcal{Y}_2^* \left(\mathcal{Y}_1^* \mathcal{B} \mathcal{Y}_1\right) \mathcal{Y}_2 = \left(\mathcal{Y}_1 \mathcal{Y}_2\right)^* \mathcal{B} \left(\mathcal{Y}_1 \mathcal{Y}_2\right) = \mathcal{Y}^* \mathcal{B} \mathcal{Y},$$

where $\mathcal{Y} = (\mathcal{Y}_1 \mathcal{Y}_2)$. Then

$$\begin{aligned}
\mathcal{Y} = \mathcal{Y}_1 \mathcal{Y}_2 &= \begin{pmatrix} v & \mathcal{W}_1 \end{pmatrix} \mathcal{Y}_2 \\
&= \begin{pmatrix} v & \mathcal{W}_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \ldots & 0 \\ 0 & & \\ \vdots & \mathcal{W}_2 & \\ 0 & & \end{pmatrix} \\
&= \begin{pmatrix} v & \mathcal{W}_1 \mathcal{W}_2 \end{pmatrix} \\
&= \begin{pmatrix} v & \mathcal{T} \end{pmatrix},
\end{aligned}$$

where $\mathcal{T} = \mathcal{W}_1 \mathcal{W}_2$, and so

$$\mathcal{Y}^* = \begin{pmatrix} v^* \\ \mathcal{T}^* \end{pmatrix}.$$

Since $\mathcal{Y}$ is a product of unitary matrices $\mathcal{Y}_1$ and $\mathcal{Y}_2$, then it follows that $\mathcal{Y}$ is again a unitary matrix, of size $m \times m$. $\mathcal{T}$ is an $m \times (m-1)$ submatrix. Using the fact that $\mathcal{Y}$ is unitary, we have that

$$\mathcal{Y} \mathcal{Y}^* = \begin{pmatrix} v & \mathcal{T} \end{pmatrix} \begin{pmatrix} v^* \\ \mathcal{T}^* \end{pmatrix} = v v^* + \mathcal{T} \mathcal{T}^* = \mathcal{I}_m \tag{34}$$

and

$$\mathcal{Y}^* \mathcal{Y} = \begin{pmatrix} v^* \\ \mathcal{T}^* \end{pmatrix} \begin{pmatrix} v & \mathcal{T} \end{pmatrix} = \begin{pmatrix} v^* v & v^* \mathcal{T} \\ \mathcal{T}^* v & \mathcal{T}^* \mathcal{T} \end{pmatrix} = \begin{pmatrix} 1 & v^* \mathcal{W}_1 \mathcal{W}_2 \\ \mathcal{W}_2^* \mathcal{W}_1^* v & \mathcal{W}_2^* \mathcal{W}_1^* \mathcal{W}_1 \mathcal{W}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0_{1 \times (m-1)} \\ 0_{(m-1) \times 1} & \mathcal{I}_{(m-1)} \end{pmatrix},$$

where the last matrix is gotten from the fact that $v^* v = 1$, $\mathcal{W}_1$ and $\mathcal{W}_2$ are unitary, which reduces $\mathcal{W}_2^* \mathcal{W}_1^* \mathcal{W}_1 \mathcal{W}_2$ to the identity matrix of size $(m-1)$, and v is orthogonal to the vectors in $\mathcal{W}_1$.

Now we observe that

$$\mathcal{Y}^*\mathcal{B}\mathcal{Y} = \begin{pmatrix} v^* \\ \mathcal{T}^* \end{pmatrix} \mathcal{B} \begin{pmatrix} v & \mathcal{T} \end{pmatrix} = \begin{pmatrix} v^* \\ \mathcal{T}^* \end{pmatrix} \begin{pmatrix} \mathcal{B}v & \mathcal{B}\mathcal{T} \end{pmatrix} = \begin{pmatrix} v^*\mathcal{B}v & v^*\mathcal{B}\mathcal{T} \\ \mathcal{T}^*\mathcal{B}v & \mathcal{T}^*\mathcal{B}\mathcal{T} \end{pmatrix} = \begin{pmatrix} v^*\gamma_1 v & v^*\mathcal{B}\mathcal{T} \\ \mathcal{W}_2^*\mathcal{W}_1^*\gamma_1 v & \mathcal{W}_2^*\mathcal{W}_1^*\mathcal{B}\mathcal{W}_1\mathcal{W}_2 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_1 & v^*\mathcal{B}\mathcal{T} \\ \mathcal{W}_2^*\gamma_1(\mathcal{W}_1^* v) & \mathcal{W}_2^*\left(\mathcal{W}_1^*\mathcal{B}\mathcal{W}_1\right)\mathcal{W}_2 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_1 & v^*\mathcal{B}\mathcal{T} \\ 0_{(m-1)\times 1} & \mathcal{W}_2^*\hat{\mathcal{B}}\mathcal{W}_2 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_1 & v^*\mathcal{B}\mathcal{T} \\ 0_{(m-1)\times 1} & \hat{\mathcal{T}}_{\mathcal{B}} \end{pmatrix},$$

which is gotten from the fact that $\mathcal{W}_1^*\mathcal{B}\mathcal{W}_1 = \hat{\mathcal{B}}$ and $\mathcal{W}_2^*\hat{\mathcal{B}}\mathcal{W}_2 = \hat{\mathcal{T}}_{\mathcal{B}}$. This matrix is upper-triangular and realizes the set $\Gamma = \{(\gamma_1, \nu_1), (\gamma_2, \nu_2), ..., (\gamma_k, \nu_k)\}$.

Turning our attention to the $n \times n$ nonnegative matrix $\mathcal{A}$, we apply the Shur decomposition theorem again to get that

$$\mathcal{X}^*\mathcal{A}\mathcal{X} = \mathcal{T}_{\mathcal{A}},$$

where $\mathcal{X}$ is a unitary matrix and $\mathcal{T}_{\mathcal{A}}$ is upper triangular and realizes the set $\Sigma = \{(\sigma_1, \mu_1), (\sigma_2, \mu_2), ..., (\sigma_k, \mu_k)\}$, since $\mathcal{T}_{\mathcal{A}}$ is similar to $\mathcal{A}$, which realizes Σ by assumption. We may partition $\mathcal{X}$ into two block matrices, an $(n-1) \times n$ block $\mathcal{V}$ and a $1 \times n$ block $\mathcal{K}$:

$$\mathcal{X} = \begin{pmatrix} \mathcal{V} \\ \mathcal{K} \end{pmatrix},$$

which implies that

$$\mathcal{X}^* = \begin{pmatrix} \mathcal{V}^* & \mathcal{K}^* \end{pmatrix}.$$

Using the fact that $\mathcal{X}$ is unitary, we have that

$$\mathcal{X}\mathcal{X}^* = \begin{pmatrix} \mathcal{V}\mathcal{V}^* & \mathcal{V}\mathcal{K}^* \\ \mathcal{K}\mathcal{V}^* & \mathcal{K}\mathcal{K}^* \end{pmatrix} = \begin{pmatrix} \mathcal{I}_{n-1} & 0_{(n-1)\times 1} \\ 0_{1\times(n-1)} & 1 \end{pmatrix}$$

and

$$\mathcal{X}^*\mathcal{X} = \mathcal{V}^*\mathcal{V} + \mathcal{K}^*\mathcal{K} = \mathcal{I}_{n\times n}. \tag{35}$$

Now assuming that $\mathcal{A}$ has the form

$$\begin{pmatrix} \mathcal{A}_1 & a \\ b^t & \gamma_1 \end{pmatrix},$$

it follows that

$$\mathcal{T}_{\mathcal{A}} = \mathcal{X}^*\mathcal{A}\mathcal{X} = \mathcal{X}^* \begin{pmatrix} \mathcal{A}_1 & a \\ b^t & \gamma_1 \end{pmatrix} \mathcal{X}$$

$$= \begin{pmatrix} \mathcal{V}^* & \mathcal{K}^* \end{pmatrix} \begin{pmatrix} \mathcal{A}_1 & a \\ b^t & \gamma_1 \end{pmatrix} \begin{pmatrix} \mathcal{V} \\ \mathcal{K} \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{V}^*\mathcal{A}_1 + \mathcal{K}^* b^t & \mathcal{V}^* a + \mathcal{K}^*\gamma_1 \end{pmatrix} \begin{pmatrix} \mathcal{V} \\ \mathcal{K} \end{pmatrix}$$

$$= \mathcal{V}^*\mathcal{A}_1\mathcal{V} + \mathcal{K}^* b^t \mathcal{V} + \mathcal{V}^* a \mathcal{K} + \mathcal{K}^*\gamma_1\mathcal{K}. \tag{36}$$

Now we consider two matrices of size $(n+m-1)\times(n+m-1)$:

$$\mathcal{Z} = \begin{pmatrix} \mathcal{V} & 0 \\ v\mathcal{K} & \mathcal{T} \end{pmatrix}$$

and its conjugate transpose

$$\mathcal{Z}^* = \begin{pmatrix} \mathcal{V}^* & \mathcal{K}^* v^* \\ 0 & \mathcal{T}^* \end{pmatrix}.$$

Since $\mathcal{V}$ has orthonormal rows, $\mathcal{V}\mathcal{V}^* = \mathcal{I}_{(n-1)}$, and since the vectors in $\mathcal{V}$ are orthogonal to the vectors in $\mathcal{K}$, then $\mathcal{V}\mathcal{K}^* v^* = 0$ and $v\mathcal{K}\mathcal{V}^* = 0$. Moreover, $\mathcal{K}$ also has orthonormal rows, so $\mathcal{K}\mathcal{K}^* = 1$, and from equation (34) we know that $v\mathcal{K}\mathcal{K}^* v^* + \mathcal{T}\mathcal{T}^* = vv^* + \mathcal{T}\mathcal{T}^* = \mathcal{I}_m$. Thus, we have that

$$\mathcal{Z}\mathcal{Z}^* = \begin{pmatrix} \mathcal{V} & 0 \\ v\mathcal{K} & \mathcal{T} \end{pmatrix} \begin{pmatrix} \mathcal{V}^* & \mathcal{K}^* v^* \\ 0 & \mathcal{T}^* \end{pmatrix} = \begin{pmatrix} \mathcal{V}\mathcal{V}^* & \mathcal{V}\mathcal{K}^* v^* \\ v\mathcal{K}\mathcal{V}^* & v\mathcal{K}\mathcal{K}^* v^* + \mathcal{T}\mathcal{T}^* \end{pmatrix} = \begin{pmatrix} \mathcal{I}_{n-1} & 0 \\ 0 & \mathcal{I}_m \end{pmatrix}.$$

We also have

$$\mathcal{Z}^*\mathcal{Z} = \begin{pmatrix} \mathcal{V}^* & \mathcal{K}^* v^* \\ 0 & \mathcal{T}^* \end{pmatrix} \begin{pmatrix} \mathcal{V} & 0 \\ v\mathcal{K} & \mathcal{T} \end{pmatrix} = \begin{pmatrix} \mathcal{V}^*\mathcal{V} + \mathcal{K}^* v^* v\mathcal{K} & \mathcal{K}^* v^* \mathcal{T} \\ \mathcal{T}^* v\mathcal{K} & \mathcal{T}^*\mathcal{T} \end{pmatrix} = \begin{pmatrix} \mathcal{I}_n & 0 \\ 0 & \mathcal{I}_{m-1} \end{pmatrix},$$

where the last matrix is gotten from the fact that $\mathcal{V}^*\mathcal{V} + \mathcal{K}^* v^* v\mathcal{K} = \mathcal{V}^*\mathcal{V} + \mathcal{K}^*\mathcal{K} = \mathcal{I}_{n\times n}$ by equation (35), $\mathcal{T} = \mathcal{W}_1\mathcal{W}_2$ and v is orthogonal to $\mathcal{W}_1$, and $\mathcal{T}^*\mathcal{T} = \mathcal{W}_2^*\mathcal{W}_1^*\mathcal{W}_1\mathcal{W}_2 = \mathcal{I}_{m-1}$. Thus, we see that $\mathcal{Z}$ is unitary. Now we consider an $(n+m-1)\times(n+m-1)$ nonnegative matrix $\mathcal{C}$ of the form

$$\begin{pmatrix} \mathcal{A}_1 & av^t \\ vb^t & \mathcal{B} \end{pmatrix}$$

and compute $\mathcal{Z}^*\mathcal{C}\mathcal{Z}$, getting

$$\mathcal{Z}^*\mathcal{C}\mathcal{Z} = \begin{pmatrix} \mathcal{V}^* & \mathcal{K}^* v^* \\ 0 & \mathcal{T}^* \end{pmatrix} \begin{pmatrix} \mathcal{A}_1 & av^t \\ vb^t & \mathcal{B} \end{pmatrix} \begin{pmatrix} \mathcal{V} & 0 \\ v\mathcal{K} & \mathcal{T} \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{V}^*\mathcal{A}_1 + \mathcal{K}^*(v^*v)b^t & \mathcal{V}^* av^t + \mathcal{K}^* v^*\mathcal{B} \\ \mathcal{T}^* vb^t & \mathcal{T}^*\mathcal{B} \end{pmatrix} \begin{pmatrix} \mathcal{V} & 0 \\ v\mathcal{K} & \mathcal{T} \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{V}^*\mathcal{A}_1\mathcal{V} + \mathcal{K}^*b^t\mathcal{V} + \mathcal{V}^*av^tv\mathcal{K} + \mathcal{K}^*v^*\mathcal{B}v\mathcal{K} & \mathcal{V}^*av^t\mathcal{T} + \mathcal{K}^*v^*\mathcal{B}\mathcal{T} \\ \mathcal{T}^*vb^t\mathcal{V} + \mathcal{T}^*\mathcal{B}v\mathcal{K} & \mathcal{T}^*\mathcal{B}\mathcal{T} \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{V}^*\mathcal{A}_1\mathcal{V} + \mathcal{K}^*b^t\mathcal{V} + \mathcal{V}^*a\mathcal{K} + \mathcal{K}^*v^*(\gamma_1 v)\mathcal{K} & \mathcal{V}^*av^t\mathcal{T} + \mathcal{K}^*v^*\mathcal{B}\mathcal{T} \\ \mathcal{T}^*vb^t\mathcal{V} + \mathcal{T}^*\gamma_1 v\mathcal{K} & \mathcal{T}^*\mathcal{B}\mathcal{T} \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{V}^*\mathcal{A}_1\mathcal{V} + \mathcal{K}^*b^t\mathcal{V} + \mathcal{V}^*a\mathcal{K} + \mathcal{K}^*\gamma_1\mathcal{K} & \mathcal{V}^*av^t\mathcal{T} + \mathcal{K}^*v^*\mathcal{B}\mathcal{T} \\ \mathcal{T}^*vb^t\mathcal{V} + \mathcal{T}^*\gamma_1 v\mathcal{K} & \mathcal{T}^*\mathcal{B}\mathcal{T} \end{pmatrix}. \tag{37}$$

Now using equation (36), the upper left corner is just $\mathcal{T}_{\mathcal{A}}$, and the lower right corner simplifies to

$$\mathcal{T}^*\mathcal{B}\mathcal{T} = \mathcal{W}_2^*(\mathcal{W}_1^*\mathcal{B}\mathcal{W}_1)\mathcal{W}_2 = \mathcal{W}_2^*\hat{\mathcal{B}}\mathcal{W}_2 = \hat{\mathcal{T}}_{\mathcal{B}}.$$

Moreover, as $\mathcal{T}^* = \mathcal{W}_2^*\mathcal{W}_1^*$ and v is orthogonal to $\mathcal{W}_1$, then the lower left corner reduces to the zero matrix of size $(m-1)\times n$, and so $\mathcal{Z}^*\mathcal{C}\mathcal{Z}$ reduces to:

$$\mathcal{Z}^*\mathcal{C}\mathcal{Z} = \begin{pmatrix} \mathcal{T}_{\mathcal{A}} & \mathcal{V}^*av^t\mathcal{T} + \mathcal{K}^*v^*\mathcal{B}\mathcal{T} \\ 0_{(m-1)\times n} & \hat{\mathcal{T}}_{\mathcal{B}} \end{pmatrix},$$

which is upper-triangular, and so its eigenvalues are just the eigenvalues of the blocks $\mathcal{T}_{\mathcal{A}}$ and $\hat{\mathcal{T}}_{\mathcal{B}}$. Since $\mathcal{T}_{\mathcal{A}}$ realizes $\Sigma = \{(\sigma_1,\mu_1),(\sigma_2,\mu_2),...,(\sigma_k,\mu_k)\}$ and $\hat{\mathcal{T}}_{\mathcal{B}}$ realizes $\Gamma' = \{(\gamma_1,\nu_1 - 1),(\gamma_2,\nu_2),...,(\gamma_k,\nu_k)\}$, it follows that $\mathcal{Z}^*\mathcal{C}\mathcal{Z}$ realizes the set

$$\Sigma\cup\Gamma' = \{(\sigma_1,\mu_1),(\sigma_2,\mu_2),...,(\sigma_k,\mu_k)\}\bigcup\{(\gamma_1,\nu_1-1),(\gamma_2,\nu_2),...,(\gamma_k,\nu_k)\}.$$

Finally, as $\mathcal{Z}^*\mathcal{C}\mathcal{Z}$ is similar to $\mathcal{C}$, we must have that

$$\mathcal{C} = \begin{pmatrix} \mathcal{A}_1 & av^t \\ vb^t & \mathcal{B} \end{pmatrix}$$

realizes the set $\Sigma \cup \Gamma'$, and is nonnegative by assumption, and this completes the proof. $\square$

Theorem 3.5 is the main tool that we will use now to construct nonnegative matrices that solve the RNIEP. Recall that, by the Perron-Frobenius theorem, nonnegative matrices have a dominant eigenvalue σ_1 that is nonnegative. Now with prescribed eigenvalues $\sigma_j \in \mathbb{R}$, relabeling the indices as needed, we order the eigenvalues

$$\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_j \geq 0 \geq \sigma_{j+1} \geq ... \geq \sigma_k$$

and assume that

$$\sigma_1 + \sigma_2 + \sigma_3 + ... = \sum_{j=1}^{k} \sigma_j \geq 0.$$

Remark. It is worth noting that these conditions are weaker than that of Suleimanova's [7], which required $(n-1)$ negative eigenvalues and precisely one positive eigenvalue.

Step 1: We start by building nonnegative matrices of size 2×2, which by Theorem 2.1, can only have real eigenvalues. We consider two cases:

Case 1: Suppose that $\sigma_1 \geq 0$ and $\sigma_2 \geq 0$.

The diagonal matrix

$$\mathcal{A} = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_1 \end{pmatrix}$$

realizes $\Sigma = \{(\sigma_1, 1), (\sigma_2, 1)\}$ and is nonnegative.

Case 2: Suppose that $\sigma_1 \geq 0$ and $\sigma_2 < 0$.

It is easy to check that the matrix

$$\mathcal{A} = \begin{pmatrix} 0 & \sqrt{-\sigma_2\sigma_1} \\ \sqrt{-\sigma_2\sigma_1} & \sigma_1 + \sigma_2 \end{pmatrix}$$

realizes $\Sigma = \{(\sigma_1, 1), (\sigma_2, 1)\}$ and is nonnegative.

Step 2: We then construct nonnegative matrices of size 3×3, using the 2×2 nonnegative matrices previously constructed; we want to choose 2×2 nonnegative matrices $\mathcal{B}_i$ that have eigenvalues σ_3 and $\mathcal{A}(2,2)$, where $\mathcal{A}(i,j)$ is the (i,j) entry of $\mathcal{A}$, chosen suitably depending on the sign of σ_2. The idea is to make the lower right corner $\mathcal{A}(2,2)$ of $\mathcal{A}$ the dominant eigenvalue of the matrix $\mathcal{B}_i$, so that $\mathcal{A}$ will have the required form for Theorem 3.5 to be applicable. We consider three cases:

Case 1: Suppose that $\sigma_1 \geq 0$, $\sigma_2 \geq 0$, $\sigma_3 \geq 0$.

A nonnegative realizing matrix is obvious here, but nevertheless we start with the algorithm here to give motivation. Using the 2×2 nonnegative matrices previously constructed, since $\sigma_2 \geq 0$, we choose

$$\mathcal{A} = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_1 \end{pmatrix}$$

and let

$$\mathcal{B}_1 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_1 \end{pmatrix},$$

so that $\mathcal{A}$ realizes $\{(\sigma_1, 1), (\sigma_2, 1)\}$ and $\mathcal{B}_1$ realizes $\{(\sigma_1, 1), (\sigma_3, 1)\}$. Note that the lower right corner $\mathcal{A}(2,2)$ of $\mathcal{A}$ is the dominant eigenvalue of $\mathcal{B}_1$, which is required in order for Theorem 3.5 to be applicable. Now we use $\mathcal{A}$ and $\mathcal{B}_1$, and choose a normalized, nonnegative eigenvector v_1 corresponding to the dominant eigenvalue of $\mathcal{B}_1$, which is possible by the Perron-Frobenius theorem, to construct

$$\mathcal{C}_1 = \begin{pmatrix} \sigma_2 & \mathcal{A}(1,2)v_1^t \\ v_1\mathcal{A}(1,2) & \mathcal{B}_1 \end{pmatrix},$$

which is of size 3×3. It is clear that $\mathcal{C}_1$ is nonnegative and in the required form for Theorem 3.5 to be applicable; now we can apply Theorem 3.5, and it follows that $\mathcal{C}_1$ realizes $\{(\sigma_1, 1), (\sigma_2, 1), (\sigma_3, 1)\}$. Of course, since the entry $\mathcal{A}(1,2)$ is zero, $\mathcal{C}_1$ is the diagonal matrix:

$$\mathcal{C}_1 = \begin{pmatrix} \sigma_2 & \mathcal{A}(1,2)v_1^t \\ v_1\mathcal{A}(1,2) & \mathcal{B}_1 \end{pmatrix} = \begin{pmatrix} \sigma_2 & 0 & 0 \\ 0 & \sigma_3 & 0 \\ 0 & 0 & \sigma_1 \end{pmatrix}.$$

The case where $\sigma_1 \geq 0$, $\sigma_2 < 0$, and $\sigma_3 \geq 0$ is not possible, since $\sigma_1 \geq \sigma_2 \geq \sigma_3$, by assumption.

Case 2: Suppose that $\sigma_1 \geq 0$, $\sigma_2 \geq 0$, $\sigma_3 < 0$.

Using the 2×2 nonnegative matrices previously constructed, since $\sigma_2 \geq 0$, we choose

$$\mathcal{A} = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_1 \end{pmatrix}$$

and let

$$\mathcal{B}_1 = \begin{pmatrix} 0 & \sqrt{-\sigma_3\sigma_1} \\ \sqrt{-\sigma_3\sigma_1} & \sigma_3 + \sigma_1 \end{pmatrix},$$

so that $\mathcal{A}$ realizes $\{(\sigma_1, 1), (\sigma_2, 1)\}$ and $\mathcal{B}_1$ realizes $\{(\sigma_1, 1), (\sigma_3, 1)\}$. The lower right corner $\mathcal{A}(2,2)$ of $\mathcal{A}$ is the dominant eigenvalue of $\mathcal{B}_1$, which is required in order for Theorem 3.5 to be applicable. Now we use $\mathcal{A}$ and $\mathcal{B}_1$, and choose a normalized, nonnegative eigenvector v_1 corresponding to the dominant eigenvalue of $\mathcal{B}_1$, which is possible by the Perron-Frobenius theorem, to construct

$$\mathcal{C}_1 = \begin{pmatrix} \sigma_2 & \mathcal{A}(1,2)v_1^t \\ v_1\mathcal{A}(1,2) & \mathcal{B}_1 \end{pmatrix} = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \mathcal{B}_1 \end{pmatrix},$$

which is of size 3×3. $\mathcal{C}_1$ is clearly nonnegative and in the form required for Theorem 3.5 to be applicable. Now we can apply Theorem 3.5, and it follows that

$$\mathcal{C}_1 = \begin{pmatrix} \sigma_2 & 0 & 0 \\ 0 & 0 & \sqrt{-\sigma_3\sigma_1} \\ 0 & \sqrt{-\sigma_3\sigma_1} & \sigma_3+\sigma_1 \end{pmatrix}$$

is nonnegative and realizes $\{(\sigma_1, 1), (\sigma_2, 1), (\sigma_3, 1)\}$.

Case 3: Suppose $\sigma_1 \geq 0$, $\sigma_2 < 0$, $\sigma_3 < 0$.

Using the 2×2 nonnegative matrices previously constructed, since $\sigma_2 < 0$, we choose

$$\mathcal{A} = \begin{pmatrix} 0 & \sqrt{-\sigma_2\sigma_1} \\ \sqrt{-\sigma_2\sigma_1} & \sigma_1+\sigma_2 \end{pmatrix}$$

and let

$$\mathcal{B}_1 = \begin{pmatrix} 0 & \sqrt{-\sigma_3(\sigma_1+\sigma_2)} \\ \sqrt{-\sigma_3(\sigma_1+\sigma_2)} & \sigma_3+(\sigma_1+\sigma_2) \end{pmatrix},$$

so that $\mathcal{A}$ realizes $\{(\sigma_1, 1), (\sigma_2, 1)\}$ and $\mathcal{B}_1$ realizes $\{(\sigma_1 + \sigma_2), 1), (\sigma_3, 1)\}$. The lower right corner $\mathcal{A}(2, 2)$ of $\mathcal{A}$ is the dominant eigenvalue of $\mathcal{B}_1$, which is required in order for Theorem 3.5 to be applicable. Now we use $\mathcal{A}$ and $\mathcal{B}_1$, and choose a normalized, nonnegative eigenvector v_1 corresponding to the dominant eigenvalue of $\mathcal{B}_1$, which is possible by the Perron-Frobenius theorem, to construct

$$\mathcal{C}_1 = \begin{pmatrix} 0 & \sqrt{-\sigma_2\sigma_1}v_1^t \\ v_1\sqrt{-\sigma_2\sigma_1} & \mathcal{B}_1 \end{pmatrix},$$

which is of size 3×3. $\mathcal{C}_1$ is clearly nonnegative and in the form required for Theorem 3.5 to be applicable; now we can apply Theorem 3.5, and it follows that

$$\mathcal{C}_1 = \begin{pmatrix} 0 & \sqrt{-\sigma_2\sigma_1}v_1^{(1)} & \sqrt{-\sigma_2\sigma_1}v_1^{(2)} \\ v_1^{(1)}\sqrt{-\sigma_2\sigma_1} & 0 & \sqrt{-\sigma_3(\sigma_1+\sigma_2)} \\ v_1^{(2)}\sqrt{-\sigma_2\sigma_1} & \sqrt{-\sigma_3(\sigma_1+\sigma_2)} & \sigma_3+(\sigma_1+\sigma_2) \end{pmatrix},$$

where $v_1^{(i)}$ is the ith component of v_1, is nonnegative and realizes $\{(\sigma_1,1),(\sigma_2,1),(\sigma_3,1)\}$.

Step 3: Now we can proceed recursively. To construct 4×4 nonnegative matrices, we start by first redefining the matrices $\mathcal{C}_1 = \mathcal{A}$ and then making its lower right corner $\mathcal{A}(3,3)$ the dominant eigenvalue of a matrix $\mathcal{B}_2$, which needs to be nonnegative and have eigenvalues σ_4 and $\mathcal{A}(3,3)$. This puts $\mathcal{A}$ in the form required to apply Theorem 3.5, and we can build the 4×4 nonnegative matrices using the information from $\mathcal{A}$ and $\mathcal{B}_2$. We consider four cases:

Case 1: Suppose that $\sigma_1 \geq 0$, $\sigma_2 \geq 0$, $\sigma_3 \geq 0$, $\sigma_4 \geq 0$.

We choose $\mathcal{C}_1$ suitably and redefine

$$\mathcal{A} = \mathcal{C}_1 = \begin{pmatrix} \sigma_2 & \mathcal{A}(1,2)v_1^t \\ v_1\mathcal{A}(1,2) & \mathcal{B}_1 \end{pmatrix} = \begin{pmatrix} \sigma_2 & 0 & 0 \\ 0 & \sigma_3 & 0 \\ 0 & 0 & \sigma_1 \end{pmatrix},$$

and let

$$\mathcal{B}_2 = \begin{pmatrix} \sigma_4 & 0 \\ 0 & \sigma_1 \end{pmatrix},$$

so that $\mathcal{A}(3,3)$ is the dominant eigenvalue of $\mathcal{B}_2$, and $\mathcal{B}_2$ has σ_4 and $\mathcal{A}(3,3)$ as its eigenvalues. Now we construct

$$\mathcal{C}_2 = \begin{pmatrix} \mathcal{A}_{2\times 2} & \mathcal{A}(1:2,3)v_2^t \\ v_2\mathcal{A}(1:2,3) & \mathcal{B}_2 \end{pmatrix},$$

where $\mathcal{A}(1:2,3)$ is the column vector $(\mathcal{A}(1,3),\mathcal{A}(2,3))^t$, v_2 is a normalized, nonnegative eigenvector corresponding to the dominant eigenvalue of $\mathcal{B}_2$, and $\mathcal{A}_{2\times2}$ is the 2×2 block on the upper left corner of $\mathcal{A}$. Clearly, $\mathcal{C}_2$ is nonnegative, and since all the assumptions of Theorem 3.5 are satisfied, we may apply the theorem - and it is obvious that

$$\mathcal{C}_2 = \begin{pmatrix} \sigma_2 & 0 & 0 & 0 \\ 0 & \sigma_3 & 0 & 0 \\ 0 & 0 & \sigma_4 & 0 \\ 0 & 0 & 0 & \sigma_1 \end{pmatrix}$$

realizes $\{(\sigma_1,1),(\sigma_2,1),(\sigma_3,1),(\sigma_4,1)\}$.

Case 2: Suppose that $\sigma_1 \geq 0$, $\sigma_2 \geq 0$, $\sigma_3 \geq 0$, $\sigma_4 < 0$.

Choosing $\mathcal{C}_1$ suitably, we redefine

$$\mathcal{A} = \mathcal{C}_1 = \begin{pmatrix} \sigma_2 & \mathcal{A}(1,2)v_1^t \\ v_1\mathcal{A}(1,2) & \mathcal{B}_1 \end{pmatrix} = \begin{pmatrix} \sigma_2 & 0 & 0 \\ 0 & \sigma_3 & 0 \\ 0 & 0 & \sigma_1 \end{pmatrix}.$$

and let

$$\mathcal{B}_2 = \begin{pmatrix} 0 & \sqrt{-\sigma_4\sigma_1} \\ \sqrt{-\sigma_4\sigma_1} & \sigma_4+\sigma_1 \end{pmatrix},$$

so that $\mathcal{A}(3,3)$ is the dominant eigenvalue of $\mathcal{B}_2$, and $\mathcal{B}_2$ has σ_4 and $\mathcal{A}(3,3)$ as its eigenvalues. Now we construct

$$\mathcal{C}_2 = \begin{pmatrix} \mathcal{A}_{2\times2} & \mathcal{A}(1:2,3)v_2^t \\ v_2\mathcal{A}(1:2,3)^t & \mathcal{B}_2 \end{pmatrix},$$

where $\mathcal{A}(1:2,3)$ is the column vector $(\mathcal{A}(1,3),\mathcal{A}(2,3))^t$, v_2 is a normalized, nonnegative eigenvector corresponding to the dominant eigenvalue of $\mathcal{B}_2$, and $\mathcal{A}_{2\times 2}$ is the 2×2 block on the upper left corner of $\mathcal{A}$. By inspection, $\mathcal{C}_2$ is nonnegative. Since all the assumptions of Theorem 3.5 are satisfied, we may apply the theorem, and thus

$$\mathcal{C}_2 = \begin{pmatrix} \sigma_2 & 0 & 0 & 0 \\ 0 & \sigma_3 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{-\sigma_4\sigma_1} \\ 0 & 0 & \sqrt{-\sigma_4\sigma_1} & \sigma_4+\sigma_1 \end{pmatrix}$$

is nonnegative and realizes $\{(\sigma_1,1),(\sigma_2,1),(\sigma_3,1),(\sigma_4,1)\}$.

Case 3: Suppose that $\sigma_1 \geq 0$, $\sigma_2 \geq 0$, $\sigma_3 < 0$, $\sigma_4 < 0$.

Choosing $\mathcal{C}_1$ suitably, we redefine

$$\mathcal{A} = \mathcal{C}_1 = \begin{pmatrix} \sigma_2 & 0 & 0 \\ 0 & 0 & \sqrt{-\sigma_3\sigma_1} \\ 0 & \sqrt{-\sigma_3\sigma_1} & \sigma_3+\sigma_1 \end{pmatrix}$$

and let

$$\mathcal{B}_2 = \begin{pmatrix} 0 & \sqrt{-\sigma_4(\sigma_3+\sigma_1)} \\ \sqrt{-\sigma_4(\sigma_3+\sigma_1)} & \sigma_4+(\sigma_3+\sigma_1) \end{pmatrix},$$

so that $\mathcal{A}(3,3)$ is the dominant eigenvalue of $\mathcal{B}_2$, and $\mathcal{B}_2$ has σ_4 and $\mathcal{A}(3,3)$ as its eigenvalues. Now we construct

$$\mathcal{C}_2 = \begin{pmatrix} \mathcal{A}_{2\times 2} & \mathcal{A}(1:2,3)v_2^t \\ v_2\mathcal{A}(1:2,3)^t & \mathcal{B}_2 \end{pmatrix},$$

where v_2 is a normalized, nonnegative eigenvector corresponding to the dominant eigenvalue of $\mathcal{B}_2$, $\mathcal{A}(1:2,3)$ is the column vector $(\mathcal{A}(1,3),\mathcal{A}(2,3))^t$, and $\mathcal{A}_{2\times 2}$ is the 2×2 block on the upper left corner of $\mathcal{A}$. By inspection, $\mathcal{C}_2$ is nonnegative. Since all the assumptions of Theorem 3.5 are satisfied, we may apply the theorem, and it follows that

$$\mathcal{C}_2 = \begin{pmatrix} \sigma_2 & 0 & \mathcal{A}(1,3)v_2^{(1)} & \mathcal{A}(1,3)v_2^{(2)} \\ 0 & 0 & \mathcal{A}(2,3)v_2^{(1)} & \mathcal{A}(2,3)v_2^{(2)} \\ v_2^{(1)}\mathcal{A}(1,3) & v_2^{(1)}\mathcal{A}(2,3) & 0 & \sqrt{-\sigma_4(\sigma_3+\sigma_1)} \\ v_2^{(2)}\mathcal{A}(1,3) & v_2^{(2)}\mathcal{A}(2,3) & \sqrt{-\sigma_4(\sigma_3+\sigma_1)} & \sigma_4+(\sigma_3+\sigma_1) \end{pmatrix},$$

where $v_2^{(i)}$ is the ith component of v_2, is nonnegative and realizes $\{(\sigma_1,1),(\sigma_2,1),(\sigma_3,1),(\sigma_4,1)\}$.

Case 4: Suppose that $\sigma_1 \geq 0$, $\sigma_2 < 0$, $\sigma_3 < 0$, $\sigma_4 < 0$.

Choosing $\mathcal{C}_1$ suitably, we redefine

$$\mathcal{A} = \mathcal{C}_1 = \begin{pmatrix} 0 & \sqrt{-\sigma_2\sigma_1}v_1^{(1)} & \sqrt{-\sigma_2\sigma_1}v_1^{(2)} \\ v_1^{(1)}\sqrt{-\sigma_2\sigma_1} & 0 & \sqrt{-\sigma_3(\sigma_1+\sigma_2)} \\ v_1^{(2)}\sqrt{-\sigma_2\sigma_1} & \sqrt{-\sigma_3(\sigma_1+\sigma_2)} & \sigma_3+(\sigma_1+\sigma_2) \end{pmatrix},$$

and let

$$\mathcal{B}_2 = \begin{pmatrix} 0 & \sqrt{-\sigma_4(\sigma_3+(\sigma_1+\sigma_2))} \\ \sqrt{-\sigma_4(\sigma_3+(\sigma_1+\sigma_2))} & \sigma_4+(\sigma_3+(\sigma_1+\sigma_2)) \end{pmatrix},$$

so that $\mathcal{A}(3,3)$ is the dominant eigenvalue of $\mathcal{B}_2$, and $\mathcal{B}_2$ has σ_4 and $\mathcal{A}(3,3)$ as its eigenvalues. Now we construct

$$\mathcal{C}_2 = \begin{pmatrix} \mathcal{A}_{2\times 2} & \mathcal{A}(1:2,3)v_2^t \\ v_2\mathcal{A}(1:2,3)^t & \mathcal{B}_2 \end{pmatrix},$$

where v_2 is a normalized, nonnegative eigenvector corresponding to the dominant eigenvalue of $\mathcal{B}_2$, $\mathcal{A}(1:2,3)$ is the column vector $(\mathcal{A}(1,3),\mathcal{A}(2,3))^t$, and $\mathcal{A}_{2\times 2}$ is the 2×2 block on the upper left corner of $\mathcal{A}$. By inspection, $\mathcal{C}_2$ is nonnegative. Since all the assumptions of Theorem 3.5 are satisfied, we may apply the theorem, and it follows that

$$\mathcal{C}_2 = \begin{pmatrix} 0 & \sqrt{-\sigma_2\sigma_1}v_1^{(1)} & \mathcal{A}(1,3)v_2^{(1)} & \mathcal{A}(1,3)v_2^{(2)} \\ v_1^{(1)}\sqrt{-\sigma_2\sigma_1} & 0 & \mathcal{A}(2,3)v_2^{(1)} & \mathcal{A}(2,3)v_2^{(2)} \\ v_2^{(1)}\mathcal{A}(1,3) & v_2^{(1)}\mathcal{A}(2,3) & 0 & \sqrt{-\sigma_4(\sigma_3+(\sigma_1+\sigma_2))} \\ v_2^{(2)}\mathcal{A}(1,3) & v_2^{(2)}\mathcal{A}(2,3) & \sqrt{-\sigma_4(\sigma_3+(\sigma_1+\sigma_2))} & \sigma_4+(\sigma_3+(\sigma_1+\sigma_2)) \end{pmatrix},$$

where $v_1^{(i)}$ and $v_2^{(i)}$ are the ith components of v_1 and v_2, respectively, is nonnegative and realizes $\{(\sigma_1,1),(\sigma_2,1),(\sigma_3,1),(\sigma_4,1)\}$.

Step n-1: Via this recursion process, and counting multiplicity of the eigenvalues at each step, at the beginning of step $n-1$ we will have obtained nonnegative matrices $\mathcal{C}_{n-3}$ of size $(n-1)\times(n-1)$ that realize $\Sigma = \{(\sigma_1,\mu_1),(\sigma_2,\mu_2),...,(\sigma_k,\mu_k)\}$, with $\sigma_j \in \mathbb{R}$ and $\sum_{j=1}^{k}\mu_j = n-1$. Now choosing $\mathcal{C}_{n-3}$ suitably and redefining it to be $\mathcal{A}$, we construct a 2×2 nonnegative matrix $\mathcal{B}_{n-2}$ in terms of $\mathcal{A}$, so that it has eigenvalues σ_{k+1} and $\mathcal{A}(n-1,n-1)$, which is the lower right corner of $\mathcal{A}$ and also the dominant eigenvalue of $\mathcal{B}_{n-2}$. Specifically, if $\sigma_{k+1}\geq 0$, let

$$\mathcal{B}_{n-2} = \begin{pmatrix} \sigma_{k+1} & 0 \\ 0 & \mathcal{A}(n-1,n-1) \end{pmatrix},$$

and if $\sigma_{k+1} < 0$, let

$$\mathcal{B}_{n-2} = \begin{pmatrix} 0 & \sqrt{-\sigma_{k+1}\mathcal{A}(n-1,n-1)} \\ \sqrt{-\sigma_{k+1}\mathcal{A}(n-1,n-1)} & \sigma_{k+1}+\mathcal{A}(n-1,n-1) \end{pmatrix}.$$

Now using the matrices $\mathcal{A}$ and $\mathcal{B}_{n-2}$, and choosing a normalized, nonnegative eigenvector v_{n-2} corresponding to the dominant eigenvalue of $\mathcal{B}_{n-2}$, we construct an $n\times n$ matrix

$$\mathcal{C}_{n-2} = \begin{pmatrix} \mathcal{A}_{(n-2)\times(n-2)} & \mathcal{A}(1:n-2,n-1)(v_{n-2})^t \\ \\ v_{n-2}\mathcal{A}(1:n-2,n-1)^t & \mathcal{B}_{n-2} \end{pmatrix},$$

where $\mathcal{A}(1:n-2,n-1)$ is the column vector $(\mathcal{A}(1,n-1),...,(\mathcal{A}(n-2,n-1))^t$ and $\mathcal{A}_{(n-2)\times(n-2)}$ is the $(n-2)\times(n-2)$ block on the upper left corner of $\mathcal{A}$. By inspection, $\mathcal{C}_{n-2}$ is nonnegative; and since all the assumptions of Theorem 3.5 are satisfied, we may apply the theorem, and thus if $\sigma_{k+1} = \sigma_k$, then $\mathcal{C}_{n-2}$ realizes $\Sigma = \{(\sigma_1,\mu_1),(\sigma_2,\mu_2),...,(\sigma_k,\mu_k+1)\}$, or if $\sigma_{k+1} \neq \sigma_k$, then $\mathcal{C}_{n-2}$ realizes $\Sigma = \{(\sigma_1,\mu_1),(\sigma_2,\mu_2),...,(\sigma_k,\mu_k),(\sigma_{k+1},1)\}$, solving the RNIEP, as required.

Example 3.6

Given $\Sigma = \{(\sigma_1,\mu_1),(\sigma_2,\mu_2),(\sigma_3,\mu_3),(\sigma_4,\mu_4)\} = \{(4,1),(7,1),(-3,1,),(-\frac{29}{9},1)\}$, we first order the σ_j:

$$7 \geq 4 \geq 0 \geq -3 \geq -\frac{29}{9},$$

and it is clear that $\sum_{j=1}^{4}\sigma_j \geq 0$, so we may implement the algorithm.
We choose the initial nonnegative matrix as

$$\mathcal{A} = \begin{pmatrix} 4 & 0 \\ \\ 0 & 7 \end{pmatrix},$$

and let

$$\mathcal{B}_1 = \begin{pmatrix} 0 & \sqrt{(3)(7)} \\ \\ \sqrt{(3)(7)} & -3+7 \end{pmatrix},$$

so that $\mathcal{A}$ realizes $\{(4,1),(7,1)\}$ and $\mathcal{B}_1$ realizes $\{(-3,1),(7,1)\}$. Note that the lower right corner $\mathcal{A}(2,2)$ of $\mathcal{A}$ is the dominant eigenvalue of the nonnegative matrix $\mathcal{B}_1$. Using $\mathcal{A}$, $\mathcal{B}_1$, and a normalized nonnegative eigenvector v_1 corresponding to the dominant eigenvalue of $\mathcal{B}_1$, we construct a 3×3 nonnegative matrix

$$\mathcal{C}_1 = \begin{pmatrix} 4 & \mathcal{A}(1,2)v_1^{(1)} & \mathcal{A}(1,2)v_1^{(2)} \\ v_1^{(1)}\mathcal{A}(1,2) & 0 & \sqrt{(3)(7)} \\ v_1^{(2)}\mathcal{A}(1,2) & \sqrt{(3)(7)} & -3+7 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & \sqrt{21} \\ 0 & \sqrt{21} & 4 \end{pmatrix},$$

which realizes $\{(4,1),(7,1),(-3,1)\}$.

Now we redefine

$$\mathcal{A} = \mathcal{C}_1 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & \sqrt{21} \\ 0 & \sqrt{21} & 4 \end{pmatrix},$$

and let

$$\mathcal{B}_2 = \begin{pmatrix} 0 & \sqrt{\frac{29}{9}(4)} \\ \sqrt{\frac{29}{9}(4)} & -\frac{29}{9}+4 \end{pmatrix},$$

so that $\mathcal{B}_2$ is nonnegative and realizes $\{(-\frac{29}{9},1),(4,1)\}$. In particular, $\mathcal{B}_2$ has the lower right corner $\mathcal{A}(3,3)$ of $\mathcal{A}$ as its dominant eigenvalue. A normalized nonnegative eigenvector v_2 corresponding to the dominant eigenvalue of $\mathcal{B}_2$ is $\left(\sqrt{\frac{29}{65}}, \frac{6}{\sqrt{65}}\right)^t$. Using $\mathcal{A}$, $\mathcal{B}_2$, $\left(\sqrt{\frac{29}{65}}, \frac{6}{\sqrt{65}}\right)^t$, and the column vector $\mathcal{A}(1:2,3) = \left(0, \sqrt{21}\right)^t$, we construct a 4×4 nonnegative matrix

$$\mathcal{C}_2 = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{21}\sqrt{\frac{29}{65}} & \sqrt{21}\frac{6}{\sqrt{65}} \\ 0 & \sqrt{\frac{29}{65}}\sqrt{21} & 0 & \sqrt{\frac{29}{9}(4)} \\ 0 & \frac{6}{\sqrt{65}}\sqrt{21} & \sqrt{\frac{29}{9}(4)} & -\frac{29}{9}+4 \end{pmatrix}$$

$$
= \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{609}{65}} & 6\sqrt{\frac{21}{65}} \\ 0 & \sqrt{\frac{609}{65}} & 0 & \frac{2\sqrt{29}}{3} \\ 0 & 6\sqrt{\frac{21}{65}} & \frac{2\sqrt{29}}{3} & \frac{7}{9} \end{pmatrix},
$$

and it is easy to check that $\mathcal{C}_2$ realizes $\Sigma = \{(4,1),(7,1),(-3,1,),(-\frac{29}{9},1)\}$, as required.

4. Solvability of the NIEP, for $n = 3$

Next, we study the solvability of the NIEP for the case $n = 3$; this result is due to Loewy and London [2].

Theorem 4.1. *Given a set $\Sigma = \{(\sigma_1,\mu_1),(\sigma_2,\mu_2),...,(\sigma_k,\mu_k)\}$, with $\sigma_j \in \mathbb{C}$ and $\sum_{j=1}^{k}\mu_j = 3$, there exists $\mathcal{A} \in \mathcal{M}_3(\mathbb{R}^+)$ that realizes Σ if and only if either $\sigma_1,\sigma_2,\sigma_3 \in \mathbb{R}$ or (up to ordering) $\sigma_1 \in \mathbb{R}$, σ_2 and σ_3 is a complex conjugate pair, and such that the following conditions hold:*

(i) $\left(\max_{1\le j\le 3}|\sigma_j|, \mu_{j_{\max}}\right) \in \Sigma$

(ii) for every pair (σ_j,μ_j) in Σ, the pair $(\bar{\sigma}_j,\mu_j)$ is also in Σ

(iii) $s_1(\Sigma) = tr(\mathcal{A}) = \sigma_1+\sigma_2+\sigma_3 \ge 0$

(iv) $[s_1(\Sigma)]^2 \le 3s_2(\Sigma)$

Proof. First, suppose we have conditions (i)-(iv) and that $\Sigma = \{(\rho,1),(re^{i\theta},1),(re^{-i\theta},1)\}$, where $0 < \theta < \pi$ and $0 < r \le \rho$ (by the Perron-Frobenius theorem). Without loss of generality, we may assume that $\Sigma = \{(\rho,1),(e^{i\theta},1),(e^{-i\theta},1)\}$. Then the Perron-Frobenius theorem tells us that $\rho \ge 1$. Consider the all-ones matrix

$$
\mathcal{T} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},
$$

which is symmetric and thus, by the spectral theorem, is diagonalizable by an orthogonal matrix; one possible diagonalizing $\mathcal{U}$ is given by

$$\mathcal{U} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & \sqrt{3} & -1 \\ \sqrt{2} & 0 & 2 \\ \sqrt{2} & -\sqrt{3} & -1 \end{pmatrix}, \tag{38}$$

and we have that

$$\mathcal{U}^t \mathcal{T} \mathcal{U} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We will use $\mathcal{T}$ to help construct $\mathcal{A}$. First, the matrix

$$\mathcal{A}' = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix},$$

by Theorem 1.1, realizes Σ; however, this matrix is not nonnegative. Instead, we first decompose the matrix into a sum of a diagonal matrix and a block-diagonal matrix containing a 2×2 rotation block:

$$\mathcal{A}' = \begin{pmatrix} \rho & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$
= \frac{\rho}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}.
$$

Now using (38), we have that

$$
\mathcal{U}\mathcal{A}'\mathcal{U}^t = \mathcal{U}\left(\frac{\rho}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}\right)\mathcal{U}^t
$$

$$
= \frac{\rho}{3}\mathcal{U}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\mathcal{U}^t + \mathcal{U}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}\mathcal{U}^t
$$

$$
= \frac{\rho}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{2} & \sqrt{3} & -1 \\ \sqrt{2} & 0 & 2 \\ \sqrt{2} & -\sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}\mathcal{U}^t
$$

$$
= \frac{\rho}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \frac{\cos(\theta)}{2\sqrt{3}} + \frac{\sin(\theta)}{6} & \frac{\sin(\theta)}{2\sqrt{3}} - \frac{\cos(\theta)}{6} \\ 0 & -\frac{\sin(\theta)}{3} & \frac{\cos(\theta)}{3} \\ 0 & \frac{\sin(\theta)}{6} - \frac{\cos(\theta)}{2\sqrt{3}} & -\frac{\cos(\theta)}{6} - \frac{\sin(\theta)}{2\sqrt{3}} \end{pmatrix}\mathcal{U}^t
$$

$$= \frac{\rho}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \frac{\cos(\theta)}{2\sqrt{3}} + \frac{\sin(\theta)}{6} & \frac{\sin(\theta)}{2\sqrt{3}} - \frac{\cos(\theta)}{6} \\ 0 & -\frac{\sin(\theta)}{3} & \frac{\cos(\theta)}{3} \\ 0 & \frac{\sin(\theta)}{6} - \frac{\cos(\theta)}{2\sqrt{3}} & -\frac{\cos(\theta)}{6} - \frac{\sin(\theta)}{2\sqrt{3}} \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{3} & 0 & -\sqrt{3} \\ -1 & 2 & -1 \end{pmatrix}$$

$$= \frac{\rho}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} +$$

$$\begin{pmatrix} \frac{\cos(\theta)}{6} + \sqrt{3}\left(\frac{\cos(\theta)}{2\sqrt{3}} + \frac{\sin(\theta)}{6}\right) - \frac{\sin(\theta)}{2\sqrt{3}} & 2\left(\frac{\sin(\theta)}{2\sqrt{3}} - \frac{\cos(\theta)}{6}\right) & \frac{\cos(\theta)}{6} - \sqrt{3}\left(\frac{\cos(\theta)}{2\sqrt{3}} + \frac{\sin(\theta)}{6}\right) - \frac{\sin(\theta)}{2\sqrt{3}} \\ -\frac{\cos(\theta)}{3} - \frac{\sin(\theta)}{\sqrt{3}} & \frac{2\cos(\theta)}{3} & \frac{\sin(\theta)}{\sqrt{3}} - \frac{\cos(\theta)}{3} \\ \frac{\cos(\theta)}{6} + \sqrt{3}\left(\frac{\sin(\theta)}{6} - \frac{\cos(\theta)}{2\sqrt{3}}\right) + \frac{\sin(\theta)}{2\sqrt{3}} & 2\left(\frac{-\cos(\theta)}{6} - \frac{\sin(\theta)}{2\sqrt{3}}\right) & \frac{\cos(\theta)}{6} - \sqrt{3}\left(\frac{\sin(\theta)}{6} - \frac{\cos(\theta)}{2\sqrt{3}}\right) + \frac{\sin(\theta)}{2\sqrt{3}} \end{pmatrix}.$$

Simplifying the upper-left entry of the second matrix gives

$$\begin{aligned} &\frac{\cos(\theta)}{6} + \sqrt{3}\left(\frac{\cos(\theta)}{2\sqrt{3}} + \frac{\sin(\theta)}{6}\right) - \frac{\sin(\theta)}{2\sqrt{3}} \\ &= \frac{\cos(\theta)}{6} + \frac{\cos(\theta)}{2} + \frac{\sqrt{3}\sin(\theta)}{6} - \frac{\sin(\theta)}{2\sqrt{3}} \\ &= \frac{\cos(\theta)}{6} + \frac{3\cos(\theta)}{6} + \frac{\sqrt{3}\sin(\theta)}{6} - \frac{\sqrt{3}\sin(\theta)}{6} \\ &= \frac{2}{3}\cos(\theta). \end{aligned}$$

Simplifying the (2,1) entry of the second matrix gives

$$\begin{aligned} &-\frac{\cos(\theta)}{3} - \frac{\sin(\theta)}{\sqrt{3}} \\ &= -\frac{1}{3}\left(\cos(\theta) + \sqrt{3}\sin(\theta)\right) \end{aligned}$$

$$= -\frac{1}{3}\left(2\frac{1}{2}\cos(\theta) + 2\frac{\sqrt{3}}{2}\sin(\theta)\right)$$

$$= -\frac{1}{3}\left(2\cos\left(\frac{\pi}{3}\right)\cos(\theta) + 2\sin\left(\frac{\pi}{3}\right)\sin(\theta)\right)$$

$$= -\frac{2}{3}\cos\left(\frac{\pi}{3} - \theta\right).$$

We repeat this process to express every entry of the second matrix in terms of $\cos(\theta)$, and $\mathcal{U}\mathcal{A}'\mathcal{U}^t$ simplifies to

$$\frac{\rho}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} -\cos(\theta) & \cos\left(\frac{\pi}{3}+\theta\right) & \cos\left(\frac{\pi}{3}-\theta\right) \\ \cos\left(\frac{\pi}{3}-\theta\right) & -\cos(\theta) & \cos\left(\frac{\pi}{3}+\theta\right) \\ \cos\left(\frac{\pi}{3}+\theta\right) & \cos\left(\frac{\pi}{3}-\theta\right) & -\cos(\theta) \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} \rho + 2\cos(\theta) & \rho - 2\cos\left(\frac{\pi}{3}+\theta\right) & \rho - 2\cos\left(\frac{\pi}{3}-\theta\right) \\ \rho - 2\cos\left(\frac{\pi}{3}-\theta\right) & \rho + 2\cos(\theta) & \rho - 2\cos\left(\frac{\pi}{3}+\theta\right) \\ \rho - 2\cos\left(\frac{\pi}{3}+\theta\right) & \rho - 2\cos\left(\frac{\pi}{3}-\theta\right) & \rho + 2\cos(\theta) \end{pmatrix} =: \mathcal{A}.$$

Since $\mathcal{A}$ is similar to $\mathcal{A}'$, we know that $\mathcal{A}$ also realizes Σ; so it remains to show that the entries of $\mathcal{A}$ are nonnegative. Using condition (iii) and the structure of Σ, we have that

$$\rho + 2\cos(\theta) \geq 0. \tag{39}$$

Moreover, using condition (iv), we have that

$$(\rho + 2\cos(\theta))^2 \leq 3\left(\rho^2 + e^{i2\theta} + e^{-i2\theta}\right) = 3(\rho^2 + 2\cos(2\theta)),$$

which implies that

$$3(\rho^2 + 2\cos(2\theta)) - (\rho + 2\cos(\theta))^2 \geq 0.$$

But

$$3(\rho^2 + 2\cos(2\theta)) - (\rho + 2\cos(\theta))^2 = 2(\rho^2 - 2\rho\cos(\theta) + \cos^2(\theta) - 3\sin^2(\theta)),$$

thus we have that

$$\rho^2 - 2\rho\cos(\theta) + \cos^2(\theta) - 3\sin^2(\theta) \geq 0.$$

Now since

$$-2\cos\left(\frac{\pi}{3} + \theta\right) = -\cos(\theta) + \sqrt{3}\sin(\theta)$$

and

$$-2\cos\left(\frac{\pi}{3} - \theta\right) = -\cos(\theta) - \sqrt{3}\sin(\theta),$$

we can factor the left-hand side, getting the constraint:

$$\left(\rho - 2\cos\left(\frac{\pi}{3} + \theta\right)\right)\left(\rho - 2\cos\left(\frac{\pi}{3} - \theta\right)\right) \geq 0. \tag{40}$$

As $0 < \theta < \pi$, we know that

$$-1 \leq \cos\left(\frac{\pi}{3} + \theta\right) < \frac{1}{2},$$

which implies that

$$-2 \leq 2\cos\left(\frac{\pi}{3} + \theta\right) < 1;$$

and by the Perron-Frobenius theorem, we know that $\rho \geq 1$, so the first factor on the left-hand side of (40) is strictly positive:

$$\left(\rho - 2\cos\left(\frac{\pi}{3} + \theta\right)\right) > 0, \tag{41}$$

which then forces the second factor to be nonnegative:

$$\left(\rho - 2\cos\left(\frac{\pi}{3}\pi - \theta\right)\right) \geq 0. \tag{42}$$

Putting this all together, the constraints (39), (41), and (42) force $\mathcal{A}$ to be nonnegative, as required. For the case where σ_j are all real, we may simply apply the algorithm from Section 3.

Proving the converse, suppose that there exists $\mathcal{A} \in \mathcal{M}_3(\mathbb{R}^+)$ that realizes Σ. Since its characteristic polynomial has real coefficients, any complex eigenvalues must come in conjugate pairs. And so either σ_1, σ_2 and σ_3 are all real, or (up to ordering) σ_1 is real and σ_2 and σ_3 is a complex conjugate pair. Moreover, condition(i) is true by the Perron-Frobenius theorem, condition(ii) follows from Theorem 1.1, condition(iii) is true since $\mathcal{A}$ is nonnegative (and so it has nonnegative trace), and condition(iv) follows from Theorem 3.1. $\square$

5. Conclusion

In this thesis, we gave a self-contained introduction to the NIEP and completely solve the problem for $n = 2$ and $n = 3$. We also used a theorem for combining eigenvalues of nonnegative matrices and the Perron-Frobenius theorem to develop an algorithm for solving the RNIEP for generalized n. Case $n = 4$ has been completely solved by Torre-Mayo, Abril-Raymondo, Alarcia-Estevez, Marijuan and Pisonero [11] and independently by Meehan [10]. For $n = 5$, Meehan and Laffey [12] solved the NIEP when considering trace zero; the problem for $n = 5$ for positive trace remains open. And although the NIEP remains open for $n \geq 5$, there is the remarkable result by Boyle and Handelman [8] which shows that given $\Sigma = \{(\sigma_1, \mu_1), (\sigma_2, \mu_2), ..., (\sigma_k, \mu_k)\}$, with $\sigma_j \in \mathbb{C}$ and $\sum_{j=1}^{k} \mu_j = n$, there exists $\mu_0 \in \mathbb{N}$ and $\mathcal{A} \in \mathcal{M}_{n+\mu_0}(\mathbb{R}^+)$ that realizes $\Sigma' = \{(\sigma_1, \mu_1), (\sigma_2, \mu_2), ..., (\sigma_k, \mu_k), (0, \mu_0)\}$, where μ_0 denotes the multiplicity of the number zero and is sufficiently large. Finding an optimal lower bound on the number of zeros needed to achieve realizability is currently an active area of research.

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